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Unexpected phenomena in mathematics and their connection with wrecks and catastrophes of recent years.

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Abstract

In the book unexpected results were obtained by the author: they are related to the most well-known to us (it seem) section of mathematics — to transformations of equations. This section of mathematics is studied in secondary school.

It turns out that the applications of customary conventional transformations to check the stability of technical means mathematical models that are used in all projection-constructional organizations in Russia and abroad can lead to errors and can become a cause of dangerous wrecks.

There are grounds to suppose that some of well-known wrecks that occurred in Russia in recent years were due to this cause.

In the book foundations of more specific advanced transformations that allow us to reduce wrecks and to make more precise our notions on the connection between a mathematical model and physical reality are given.

The book is dedicated to a wide range of readers: engineers, teachers in mathematics and physics, undergraduates from technical, mathematical or physical departments in high schools.

Introduction

We usually think that in mathematics everything is well known and nothing unexpected can occur. Especially this concerns elementary mathematics studied in secondary school.

After you have read this small book you will change this opinion. Quite unexpected interesting phenomena may occur in the most usual and conventional sections of mathematics — for example in the section of equations transformations.

Since secondary school we know that we can transfer terms of the equation from the rights side to its left with the change of the sign. Similarly we are in the habit to multiply and divide all terms by a number different from zero etc. Everyone is in the habit of widely applying these transformations.

But nobody notice that during these well-known transformations (since secondary school) something unexpected may occur. This state of things was preserved up to resent years.

The book is intended for engineers, undergraduates, teachers — for all who is acquainted with the most simple differential equations having constant coefficients. Just during transformations of differential equations these new interesting and quite unexpected phenomena appear. The author's aim is to describe and investigate them.

Note that in this book not only mathematical curiosities will be treated but serious practical applications that lead to wreckages and catastrophes. The book also deals with the methods of preventing them.

The author wishes that the reader would seriously think over the phenomena described in the book. Everyone can become a victim of wrecks or catastrophes. And there is a possibility of decreasing wreckages, it is necessary to use this possibility. In the book the author relates how to deal with these unexpected phenomena.

§1. Differential equations and their transformations

Later we shall speak only about the most simple differential equations having constant coefficients. They can be widely found in applications. For example, an equation

$$\frac{dx}{dt} = kx \tag{1}$$

describes population number change in a country where birth and death rates are yearly constant and do not depend on time. Coefficient k in equation (1) is proportional to the difference between births and deaths. The following function is a solution of equation (1):

$$x = c_1 e^{kt}. \tag{2}$$

We can easily see this if we introduce function (2) in equation (1). Since for function (2) we have $\dot{x} = c_1 k e^{kt}$ then after this introduction equation (1) will turn into in identity. This means that function (2) is solution. (Remind that such function is called a solution of a differential equation whose introduction into this equation turns it into an identity.)

Arbitrary constants c_1; c_2; ...; c_n enter into a general solution of a differential equation. The quantity of arbitrary constants is equal to an equation degree, i. e. to degree of derivatives entering into the equation. In order that solution be completely defined it is necessary that besides a differential equation itself initial conditions be also given, a value of function $x(t)$ and its derivatives when $t = 0$. A number of necessary initial conditions is equal to a degree of the equation. For the equation of the first degree (1) it is sufficient to have one initial condition.

For example, suppose that 1996 is an initial moment of time and that in this year the population of the country we are interested in is equal to ten millions, i. e. 10^7. Then from the formula (2) we can find that $c_1 = 10^7$ and the population of the country will increase by an exponent $x = 10^7 e^{kt}$ in the course of time.

The equation of pendulum oscillation movement can serve as an example of second degree equation:

$$\ddot{x} + a\dot{x} + bx = 0, \tag{3}$$

where x — pendulum deviation from equilibrium position, a and b — parameters depending on a pendulum inertia moment and on friction in a suspension point.

If we use a Cauchy form for differentiation operator $\frac{d}{dt} = D$ equation (3) can be written in the form:

$$(D^2 + aD + b)x = 0. \tag{4}$$

Later we shall widely use this designation for differentiation operator $D = \frac{d}{dt}$.

A general solution of equation (4) is of the form:

$$x = e^{-\frac{a}{2}t}(c_1 \sin \sqrt{b - \frac{a}{2}}t + c_2 \cos \sqrt{b - \frac{a}{2}}t). \tag{5}$$

Two arbitrary constants c_1 and c_2 determined from the initial conditions $x(0) = x_0$; $\dot{x}(0) = x_1$ enter into this solution. Values of a function itself $x(t)$ and of its derivatives $\dot{x}$ when $t = 0$ are initial conditions. Solution (5) shows that gradually extinguishing oscillations with the frequency $\sqrt{b - \frac{a}{2}t}$ depending on parameters a and b can be considered the pendulum movement law.

Differential uniform equation with constant coefficients of an arbitrary n-th degree can be written in the form

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_0)x = 0. \tag{6}$$

Its solution depends on the so-called characteristic polynomial

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0. \tag{7}$$

It is seen that in order to obtain a characteristic polynomial it is sufficient to put letter λ instead of a differentiation operator. As a result we obtain n-th degree polynomial having n roots λ_1; λ_2; $\ldots$; λ_n.

If all these roots are real and different a general solution of equation (7) will be written in the form:

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}. \tag{8}$$

If among polynomial roots there are complex roots they can enter there only by conjugated pairs:

$$\lambda_{i,i+1} = \alpha \pm j\beta, \quad j = \sqrt{-1}. \tag{9}$$

A term of the form:

$$e^{a_i t}(c_i \sin \beta t + c_{i+1} \cos \beta t). \tag{10}$$

will correspond to each pair of complex roots in a general solution.

We see that if all roots of a characteristic polynomial have negative real parts then any solution will tend to zero in the course of time at any initial conditions. If among roots of a characteristic polynomial there are multiple roots then terms of the form $c_i t^m e^{\lambda_i t}$, can appear in the solution. But a general conclusion remains unchanged: if all roots of a characteristic polynomial have negative real parts any solution of equation (6) is sure to tend to zero in the course of time.

Now let's speak about transformations of equations. For example, transfers of term from the left side to the right and vice versa with the corresponding change of sign, the division of all equation terms by the same number not equal to 0 — are the simplest transformations of equations. It is clear that during such transformations the solutions of equations do not change. In general during transformations it is necessary to use only equivalent transformations, transformations are called equivalent if during them solutions do not change. All solutions of an transformed equation must coincide with all solutions of an initial equation (see "Mathematical Encyclopedia", vol. 4, p. 800, ed. by "Sovetskaya Encyclopedia", 1984).

Besides the most simple equivalent transformations (the transfer of terms, multiplication or division by a number not equal to zero) such transformation as terms differentiation is widely used when studying differential equations. This transformations is also an equivalent transformation.

For example, let's consider an equation

$$\dot{x} + x = 0 \tag{11}$$

with an initial condition $x(0) = 0$. A general solution of equation (11) is of the form:

$$x = c_1 e^{-t}. \tag{12}$$

To an initial condition $x(0) = 0$ the only solution $x = 0$ satisfies. In particular, from it follows that $\dot{x}(0) = 0$. If we differentiate all equation (11) terms then we shall come to the equation

$$\ddot{x} + \dot{x} = 0 \tag{13}$$

The degree of the equation has increased and we must add one more initial condition for the first derivative, for $\dot{x}(0)$. Surely such a condition must not be chosen arbitrary. It is necessary to choose such a condition for $\dot{x}(0)$ that in an initial equation (11) would be satisfied automatically. We are quite certain that in an initial equation there was: $\dot{x}(0) = 0$. This equality must've regarded as the second initial condition for equation (13). The polynomial

$$\lambda^2 + \lambda \tag{14}$$

will be considered a characteristic polynomial of the equation (13) with the roots $\lambda_1 = 0$, $\lambda_2 = -1$. Thus a general solution of equation (13) is of the form:

$$x = c_1 + c_2 e^{-t}. \tag{15}$$

To initial conditions $x(0) = 0$, $\dot{x}(0) = 0$ corresponds only one solution $x = 0$, i. e. the same solution as in equation (11). It is clear that if additional initial conditions are given correctly terms differentiation becomes an equivalent transformation.

Just the same is true when we multiply the right and the left sides of a differential equation by any polynomial $B(D) = b_m D^m + \ldots + b_0$ of a differentiation operator. It will be an equivalent transformation. So if we multiply equation (11) by an operator polynomial $D + 2$ we shall obtain an equation $(D^2 + 3D + 2)x = 0$ whose characteristic polynomial has the roots $\lambda_1 = -1$, $\lambda_2 = -2$. A general solution of the equation is of the form $x = c_1 e^{-t} + c_2 e^{-2t}$. Initial condition $x(0) = 0$ and $\dot{x}(0) = 0$ satisfy only a solution $x = 0$, the picture is the same as in equation (11). We see that while multiplying by a polynomial of differentiation operator if additional initial conditions are set correctly we again deal with equivalent transformations.

Equivalent transformations are widely used when one equation is reduced to a system of equations of lower degrees or when a system of equations is reduced to one equation. So if in equation (3) we put $x = x_1$ and $\dot{x} = x_2$ then equation (3) will turn into a system of two equations of the first degree

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -bx_1 - ax_2 \end{cases} \tag{16}$$

If we differentiate the first of them we receive $\ddot{x}_1 = \dot{x}_2$. If we now put in the second equation from (16) instead of x_2 and $\ddot{x}_1$ values equal to them ($x_2 = \dot{x}_1$, $\ddot{x}_1 = \dot{x}_2$) we shall return to equation (3) in relation to $x_1 = x$.

In a general case any system of several differential equations having a constant coefficients of different degrees can be reduced to the so called Cauchy form, to the form of n equations of the first degree in relation to:

$$\left.\begin{array}{c} \dot{x}_1 = a_{11}x_1 + \ldots + a_{1n}x_n \\ \cdots \\ \dot{x}_n = a_{n1}x_1 + \ldots + a_{nn}x_n \end{array}\right\} \qquad (17)$$

System (17) is often written in a vector-matrix form

$$\dot{x} = Ax, \qquad (18)$$

where x — n-dimensional vector of variables x_1, x_2, …, x_n, A — matrix of coefficients a_{ij}.

The Cauchy form is extremely convenient in this case to calculate a characteristic polynomial. By means of these general formulas for a long period of time a good compilation of programs for calculations has been developed. Really as it is known from linear algebra a characteristic polynomial of equations (17) and (18) is equal to a determinant of a matrix $(\lambda E - A)$, where E — a unit matrix. The matrix determinant is calculated by standard programs on ECM. Therefore transformation to the Cauchy form (17) and (18) is a widely applied transformation.

§2. Stability solutions

For many practical applications it is important not only to be able to calculate a solution of the equation but to estimate the stability of this solution. Let's return to a simple equation (1) having a general solution (2). To an initial condition $x(0) = 0$ a solution $x = 0$ satisfies. But initial conditions are rarely known exactly. As a rule in initial conditions small mistakes are inevitable. If these mistakes increase in the course of time then the solution is not stable. So if we put that $x(0) = 0$ but evidently we have $x(0) = 10^{-4}$ then when $kt = 10$ a true value $x(t)$ will already not be equal to zero and $x = 2.2$. A mistake will increase exponentially and in a short period of time it will become inadmissible. Therefore the stability investigation is extremely important.

For linear systems having constant coefficients simple methods of stability checking exist (without finding solutions themselves). Really the solutions character of a linear differential equation with constant coefficients or of a system of such equations is fully determined by a characteristic polynomial. If in all roots of a characteristic polynomial real parts are negative then any solution $x(t)$ will tend to zero at $t \to \infty$. Thus difference between solutions corresponding to different initial conditions will also tend to zero and all solutions will be stable (to be more exact — they will be asymptotically stable).

In 1895 a German mathematician A. Hurvitz (1859-1919) discovered conditions to which coefficients of a polynomial (7) must correspond in order that all its roots have negative real parts. Polynomials whose roots have negative real parts are called Hurvitz polynomials.

For example, polynomials of the second degree

$$a_2\lambda^2 + a_1\lambda + a_0 \tag{19}$$

will be Hurvitz polynomials if all their coefficients are positive (to be more exact —- the eldest coefficient of a polynomial is always turned into a positive value). For polynomials of the third degree

$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \tag{20}$$

coefficients positively is already not sufficient for polynomial (20) to be Hurvitz. It is necessary and sufficient that an inequality

$$a_2a_3 > a_3a_0 \tag{21}$$

be additionally fulfilled.

For polynomials of higher than the third degree necessary and sufficient conditions are more complex. They can be found in textbooks on automatic control theory. It is convenient to use a very simple necessary (but not sufficient) condition: a polynomial of any degree can be of a Hurvitz type only when all its coefficients are positive and there are not one among them negative or the one equal to zero.

Now notice that for linear systems having constant coefficients solutions stability does not depend on initial conditions: either all solutions at any initial conditions are stable or they are not. Therefore for linear systems we often speak not about stability of solution but about the stability of a system. We call a system stable when its solutions satisfying initial conditions are stable.

In non-linear systems everything is more complex. In them a solution satisfying one initial condition can be stable and if it satisfies another initial condition — instable. Therefore for non-linear systems we can speak only about solutions stability.

The first unexpected phenomenon.

Already in the thirties the following first surprise appeared during stability investigation: although the multiplication of the right and the left sides of the equation by a polynomial of a differentiation operator $D = \frac{d}{dt}$ is an equivalent transformation such a transformation can change stability. For example, let's return to equation (11) with an initial condition $x(0) = 0$. It has a solution and this solution is stable since a characteristic polynomial of equation (11) is of the form $\lambda + 1$ and is Hurvitz. Now let's multiply equation (11) by an operator polynomial. We shall obtain an equation of a second degree:

$$(D^2 - 1)x = 0 \tag{22}$$

whose characteristic polynomial has roots $\lambda_1 = -1$, $\lambda_2 = +2$ and a general solution is of the form :

$$x = c_1 e^{-t} + c_2 e^{t} \tag{23}$$

A single solution $x = 0$ obtained from formula (23) satisfies initial conditions $x(0) = 0$, $\dot{x}(0) = 0$ when $c_1 = c_2 = 0$. This solution coincides with the solution of equation (11) and it is just right since a multiplication by an operator polynomial is an equivalent transformation. But the solution $x = 0$ of equation (22) is not stable. Really if initial condition deviate from zero by even small numbers δ_1 and δ_2 and instead of $x(0) = 0$ and $\dot{x}(0) = 0$ we have $x(0) = \delta_1$; $\dot{x}(0) = \delta_2$ then on the basis of formula (23) we shall find that $c_1 = 0.5(\delta_1 - \delta_2)$; $c_2 = 0.5(\delta_1 + \delta_2)$ and thus

$$x = 0.5(\delta_1 - \delta_2)e^{-t} + 0.5(\delta_1 + \delta_2)e^{t} \qquad (24)$$

It is clear that even if δ_1 and δ_2 are small the difference between solution (24) and a solution $x = 0$ will unlimitedly increase in the course of time.

Even this simple example shows that even after equivalent transformations the investigation of a transformed system may not give a correct answer to the question of stability. But in this case difficulties were overcome by introducing the following simple prohibition: at investigating stability the multiplication by a non-Hurvitz operator polynomial is not admissible. As we know multiplication by a Hurvitz polynomial of a differentiation operator is admissible and it does not influence the conclusion about stability.

While investigating stability preservation at variations (small changes) of parameters and coefficients of differential equations (that are in practice inevitable) we have come across more interesting and serious unexpected phenomena. Although we always speak about differential equations with constant coefficients in fact there are never found ideally constant coefficients. Coefficients of differential equations depend on parameters of an investigated system. And in any real system parameters can not remain ideally constant. Small deviations of parameters, their variations are quite inevitable.

So for example, equation (3) whose solution serves as a function (5) describes oscillations of a physical pendulum. Coefficient a depends on the moment of a pendulum inertia and thus it depends on the temperature of environs when temperature changes as a result of thermal expansion. And thus linear dimensions of a pendulum change and an inertia moment also changes. Coefficients b in equation (3) depends on a friction value at a point of suspension. But a friction coefficient depends on temperature, on materials wear off in a point of suspension. Therefore coefficient b will also undergo variations, small changes.

Therefore in practice it is not sufficient that an investigated system is simply stable. It is necessary that it preserves stability when parameters variate (as it usually occurs in practice).

When we investigate stability preservation at parameter variations quite recently we have come across upon extremely interesting mathematical unexpected phenomena.

§3. Mathematical unexpected phenomena

Let us consider a system of two differential equations

$$(D^3 + 4D^2 + 5D + 2)x_1 = (D^2 + 2D + 1)x_2 \quad (25)$$

$$(D + 1)x_2 = (D^2 + 4D + 5)x_1 \quad (26)$$

System (25)–(26) can be reduced to one equation in relation to x_1:

$$(D^3 + 5D^2 + 7D + 3)x_1 = 0 \quad (27)$$

if, for example, we exclude a variable x_2 using equivalent transformations.

A characteristic polynomial of system (25)–(26)

$$\lambda^3 + 5\lambda^2 + 7\lambda + 3 \quad (28)$$

having roots $\lambda_1 = -3$, $\lambda_2 = \lambda_3 = -1$ is Hurvitz polynomial and system (25)–(26) is stable.

A general solution of system (25)–(26) as it can be easily checked is of the form:

$$x_1 = c_1 e^{-3t} + (c_2 t + c_3)e^{-t} \quad (29)$$

This once more confirms the opinion that all solutions of system (25)–(26) satisfying any initial conditions are stable.

But system (25)–(26) can loose stability even if variations of some coefficients are infinitely small. For example, if in equation (25) a coefficient in the term D^2x_2 will be equal not to 1 but 0,999 and other coefficients will remain unchanged then a characteristic polynomial will become

$$-0,001\lambda^4 + 0,996\lambda^3 + 4,995\lambda^2 + 7\lambda + 3 \quad (30)$$

and it will not be Hurvitz any more since a sign of the coefficient in x^{4}' is contrary to a sign in other coefficients. Polynomial (30) has a large positive root $\lambda_4 = 1001$ and thus in the solution of the equation a term that is quickly increasing appears: $x_1 = c_1 e^{-3t} + (c_2 t + c_3)e^{-t} + c_4 e^{1001t}$. Stability can be as well lost if some other coefficients variations are infinity small.

Note that if a coefficient in a term D^2x_2 is not less but is larger than 1 (for example if it will be equal not 0,999 but 1,001) and other coefficients will remain unchanged then a characteristic polynomial of a system (25) will become:

$$0,001\lambda^4 + 1,004\lambda^3 + 5,005\lambda^2 + 7\lambda + 3 \quad (31)$$

and it will be Hurvitz, a system preserves stability. Therefore the following conclusion can be drawn: **Only variations of quite a definite sign lead to stability loss.**

An attentive reader can himself check the correctness of characteristic polynomials (28), (30), (31) calculation. Really for a system of two differential equations of the form

$$P_1(D)x_1 = P_2(D)x_2$$
$$P_3(D)x_2 = P_4(D)x_1$$

where $P_1(D) \ldots P_4(D)$ — polynomials of differentiation operator $D = \frac{d}{dt}$, its characteristic polynomial will be equal to a determinant

$$\pm \begin{vmatrix} P_1(D)x_1 & P_2(D)x_2 \\ P_3(D)x_4 & P_4(D)x_3 \end{vmatrix}$$

i.e. it will be equal to $\pm[P_1(D)P_3(D) - P_2(D)P_4(D)]$ (a double sign is taken as polynomial roots do not change when all terms change sign. Therefore such sign at which the majority of terms are positive is chosen. With the help of this formula we obtain the following expression for a characteristic polynomial for system (25)–(26):

$$(D^2 + 2D + 1)(D^2 + 4D + 5) - (D^3 + 4D^2 + 4D + 2)(D + 1).$$

If we carry out multiplication, reduce similar terms and put λ instead of D we obtain formula (28). But if in equation (25) a coefficient in a term D^2x_2 is not equal to 1 but 0,999 then a characteristic polynomial will be equal to

$$(0,999D^2 + 2D + 1)(D^2 + 4D + 5) - (D^3 + 4D^2 + 5D + 2)(D + 1).$$

After multiplication, reduction of similar members and change D by λ we shall obtain formula (30). But if in equation (25) a coefficient in term D^2x_2 is not equal to 1 but 1,001 then after similar calculations we shall obtain formula (31).

Now let's transform system (25)–(26) to the Cauchy form. For this it is sufficient to introduce new variables x_3 and x_4. New variables can be determined by equalities:

$$\left.\begin{array}{l} x_3 = \dot{x}_1 + 2x_1 - x_2 \\ x_4 = \dot{x}_3 \end{array}\right\} \qquad (32)$$

In relation to new variables equation (25) will turn into a system of three first degree equations, i.e. it will be in a Cauchy form:

$$\left.\begin{aligned} \dot{x}_1 &= -2x_1 + x_2 + x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_3 - 2x_4 \end{aligned}\right\} \tag{33}$$

It is easy to check the correctness of transfer from (25) to (33) by an inverse transformation. By excluding variables x_3 and x_4 from equation (33) we shall again obtain equation (25).

Once more it became evident that equations (25) and (33) are equivalent.

Now let's transform equation (26). The transformation only consist in a division of terms and their transfer from one side of the equality into another (with the change of a sign). Equivalence of such transformations does not arise any doubts. We shall obtain :

$$\Big[(D^2 + 2D)x_1 - Dx_2\Big] + [(2D + 4)x_1 - 2x_2] + x_1 + x_2 = 0. \tag{34}$$

By comparing (34) with equalities (32) we see that a variable x_4 corresponds to the first square bracket, $2x_3$ corresponds to the second square bracket. Finally equation (26) can be transformed as follows:

$$x_2 = -x_1 - 2x_3 - x_4 \tag{35}$$

It is clear that in relation to new variables equation (26) turns into an equation of a zero degree i.e. it turns into a relation between variables that does not contain any derivatives. Equations (33)–(35) are equivalent to equations (25)–(26). This can be once more checked if we calculate a characteristic polynomial of equations (33)–(35) system. As easily seen it will remain the same as in system (25)–(26). And as before it will be of the form (28). System (33)–(35) will be stable and its equivalence to system (25)–(26) does not arise any doubts.

And here we encounter with the following unexpected phenomenon: if any coefficients of system (33)–(35) undergo small changes and we check new systems for their stability then we shall invariably see that all these systems are stable.

System (33)–(35) preserves its stability at variations, (or small changes) of its any coefficients although system (25)–(26) does not possess such a property.

WE MUST UNDERLINE THAT SYSTEMS (25)–(26) AND (33)–(35) ARE EQUIVALENT BETWEEN THEMSELVES. THEIR CHARACTERISTIC POLYNOMIALS ARE IDENTICAL, SOLUTIONS SETS ARE THE SAME. BUT IN PRESERVATION STABILITY PROPERTY AT PARAMETERS VARIATIONS THEY ARE STRIKINGLY DIFFERENT. HERE IS A GREAT SURPRISE.

The discovery of this unexpected result is of great practical value. For some time past we were in the habit to judge about stability of different objects and systems and about stability preservation at small inevitable variations of parameters and their coefficients surely by transformed equations. Without equivalent transformations no investigation can be undertaken. But if stability preservation property at coefficients variations can appear and disappear during equivalent equations transformations then the following conclusion must be drawn. Conventional methods of checking stability and its preservation are not complete. They can lead to incorrect results. And an incorrect answer to such serious questions as stability can become the cause of wreckages leading to deaths of people.

§4. The explanation of an unexpected phenomenon

The possibility of changing stability property at parameters variations after equivalent transformations can be explained by an example of systems (25)–(26) and (33)–(35). For the first time such an explanation was given in Yu.P. Petrov's book "Optimal control systems synthesis when perturbing forces are incompletely known", Leningrad, University edition, 1987. Really let us analyze in details the meaning of the following statement: **"A system of equations preserves stability of its solutions at variations of (small deviations) coefficients from calculated values"**. In fact this statement does not deal with properties of an investigated system itself but with properties of a set of other systems whose coefficients differ (although not much) from coefficients of an initial investigated system. (Such a set of systems is called a surrounding of an initial system.)

Now let us recall that equivalent transformations must not change a solution of an initial system but they by no means must leave unchanged properties of its surrounding. In the definition of equivalent transformations nothing is said about this. Since the property of stability preservation at parameters variations is the property of a surrounding it can appear and disappear during equivalent transformations of equations.

Hence it follows that a system (25)–(26) and (33)–(35) are not exceptions in any particular way. There are many such systems changing some their properties during equivalent transformations. Later we shall show the whole classes of such systems. Hence it follows that conventional methods (used in all projectional and constructional bureaus in industry) of checking stability preservation by means of a characteristic polynomial or by Lyapunov function of an investigated system are not complete. Therefore these methods are not reliable and do not always lead to correct results.

In order to be sure in stability preservation at parameters variations it is necessary to introduce a new mathematical conception: the **conception of equivalence in a widened sense.** The cause is the following: usual equivalent transformations (to be more exact let us later call them transformations equivalent in a classical sense) do not guarantee the preservation of properties in the surrounding of an investigated system including the properties of stability preservation at parameters variations.

The following systems are called equivalent in a widened sense:

1. such systems that are equivalent in a classical sense (i.e. their solutions coincide)

2. surroundings of solutions differ from each other little (in particular if solutions of an investigated system are stable then they are stable in all systems that are in the surrounding of an investigated system).

Later we shall give criteria allowing to judge whether the system is equivalent in a widened sense or not.

Let us at once note that the majority of systems equivalent in a classical sense will be equivalent in a widened sense as well. Just because of this the necessity of introducing a new mathematical conception has not been comprehended for so long period of time.

In fact if in an equation (for example)

$$2\dot{x} = -2x \tag{36}$$

all solutions are stable then the following equation possesses the same property:

$$\dot{x} + x = 0 \tag{37}$$

This equation can be obtained from (36) by multiplying by 0,5 and with the help of transferring a term $-x$ into the left side. Equations (36) and (37) are equivalent between themselves in both — a classical and a widened sense. And for the majority of equations and equations systems the things will be getting on just in this way. Systems equivalent between themselves in a classical sense but not in a widened sense rarely occur. But it is quite necessary to investigate them since each occurrence with such a rare system can lead to wrecks and people's loss of life. In the next paragraph we shall discuss this problem.

§5. Practical applications

The stability checking is of great importance when we investigate automatic control systems with which the majority of objects in industry and transport is supplied now. The majority of time airplanes fly by means of autopilot. In the sea ships move by automatic steering. At the majority of factories the constancy of technological processes is achieved by automatic regulators.

In control systems variables x_1, x_2, …, x_n are (as a rule) deviations of process characteristics from a desired value. If a system of differential equations with unstable solutions is a mathematical model of an automatic control system then these deviations will quickly increase. And a normal work of the system will become impossible.

Therefore even on the stage of projection in all projection-construction bureaus the checking of a projected system is always undertaken. It is checked whether a projected system be stable or not. Only a system that is stable according to the checking calculation is allowed to be manufactured "in metal". Let us suppose that we have made a mistake in calculation and an unstable system of control was admitted to be a stable one. This fact is unpleasant but not dangerous. At tests the system instability will at once appear since increasing deviations of a real duration in a controlled process from a desired one will be present. By all means such a system will be rejected and the corresponding losses will be registered.

More dangerous is a mistake in a problem of stability preservation at variations of parameters. Let us suppose that we investigate a control system whose mathematical model are equations (25)–(26). For some time past at calculations a quick-calculating technique is most often applied. For the convenience of using standard programs an investigated mathematical model is reduced to a standard form, to Cauchy form, to, i.e. - a system of first degree equations. By reducing a system (25)–(26) to Cauchy form we shall come to well-known equations (33)–(35). At investigating these equations we come to the conclusion that in variations of any coefficients a system (33)–(35) will preserve stability. But an initial system (25)–(26) (as it was shown in §3 at variations of some coefficients and only at variations of a certain sign) looses stability. Here is the explanation: systems (25)–(26) and (33)–(35) are equivalent to each other in a classical sense but not in a widened sense. If a new mathematical conception — the equivalence in a widened sense — is not introduced or if we limit ourselves by a classical conception of equivalent systems then we must admit that our projected system of an automatic control (after having calculations in equations (33)–(35)) is stable and preserving stability at parameters variations. And therefore we must give recommendations on its manufacture "in metal".

Since at manufacturing small deviations of real parameters in any object from calculated values are inevitable and a sign of these deviations is not predicted then at manufacturing we can obtain a control system whose mathematical model will not be equations (25)–(26) but an equation that differ from them very little:

$$(D^3 + 4D^2 + 5D + 2)x_1 = (1,001D^2 + 2D + 1)x_2 \tag{38}$$

$$(D + 1)x_2 = (D^2 + 4D + 5)x_1, \tag{39}$$

Here only one coefficient has deviated from a calculated value by 1/1000. A characteristic polynomial of a system (38)–(39) (as it was indicated in §3) is of the form (31) and is of a Hurvitz type. Thus a really manufactured system is stable and tests of a manufactured system will positively confirm it.

Since a manufactured control system has successfully undergone tests it can be mounted on an object and this system can work without defects for a long period of time. But during exploitation a small drift of all parameters and coefficients of a mathematical model is inevitable. It is due to the wear out and the aging of working components. But it is quite possible that at some unexpected moment a coefficient in D^2x_2 instead of an initial value 0,001 will become 0,999. At this moment a characteristic polynomial of the system (as it was shown in §3) will become equal to a polynomial (30) and it will cease to be a Hurvitz polynomial. And this means that a real system will loose stability. The loss of stability at an unexpected moment of time by itself creates a wreck situation. And if the means of defense do not work this situation can lead to a dangerous wreck that is connected with people's deaths.

We can indicate the following cause of the wreck: a system (33)–(35) is equivalent to a system (25)–(26) in a classical sense but not in a widened sense as the surrounding of these systems are different. We see that if the difference between the conception of equivalence in a classical and in a widened sense is ignored, each encounter with such system for which these conceptions do not coincide can become deathly dangerous.

At the same time the analysis of examples with systems (25)–(26) and (33)–(35) shows why there was no need in a new mathematical conception up to recent years. And no wrecks connected with mistakes made at calculations of stability preservation occurred at that time as well.

First of all dangerous phenomena appear only in systems of a rather high degree (not less than the third degree). Besides in systems of the third-fifth degrees these phenomena appear in a rather simple form and can be easily depicted. An experienced engineer will at once notice dangerous properties of a system (25)–(26) and will not allow it to be manufactured. An example with the system (25)–(26) was given only in order that the reader can watch all transformations. If the reader wishes he can repeat them and make sure that there is no mistake or any trick.

But in modern technique we have to deal with control systems up to the 20–40 degrees. Here no experience, no intuition of an engineer will help. In order to prevent wrecks it is necessary to apply more advanced modern mathematical methods. It is also necessary to take into account differences between equivalence in a classical and in a widened sense.

Secondly the possibility of stability loss at parameters variations in rather simple control systems was made apparent during tests by simple "swimming" of all parameters and coefficients. But in modern systems of high degrees this old method does not work. The main difficulty lies in the following: stability loss can occur at variations combinations of different parameters with a different sign. For example, stability loss can take place in a certain system only when the first parameter deviates from a calculated value to a positive side, the second parameter — surely — to a negative side, the third — again to a positive side etc. If the behavior of an investigated system depends on N parameters then the number of combinations in positive and negative variations is equal to 2^N and they all must be checked. If N is=40 (and for modern complex systems it is a very moderate number) then the number of combinations that must be checked is equal to 2^{40} and this is more than 10^{12}. It is clear that even if we apply the most quick-operating calculation technique such a number of checks can not be undertaken. And we must by all means transfer to a more advanced calculation and projection methods that apply a new mathematical conception of equivalence in a widened sense. Otherwise wrecks will be inevitable.

Thirdly, up to now the stability calculation of control systems was as a rule made by means of "real exits" (about them we shall speak in §7), without a transfer to the Cauchy form. It became necessary to transfer to the Cauchy form everywhere only during mass application of a quick-operating calculation technique for the calculations and standard programs, during the transfer to automated projection.

Such a transfer is by all means progressive but it must be combined with revision and making more exact the applied mathematical apparatus. The ignorance of difference between an equivalence in a classical and a widened senses is not dangerous for hand calculation. But it can become extremely dangerous during automated projection. The automated projection enters more and more into the practice of our life, into the calculations of the majority of projecting-constructional bureaus in industry in recent time.

§6. Wrecks and catastrophes

Have you come across such wrecks whose cause was conventional projection methods that do not take into account the difference between equivalence in a classical and in a widened senses? It was already shown that non-critical application of conventional methods (without additional checks) could lead to appearance of dangerous systems that might loose stability at small parameters drifts. These drifts are quite inevitable in practice (at an unpredictable moment of time). Really, have you encountered wrecks that have this cause ?

It is difficult to give a trustworthy answer to this question since the investigation of wrecks causes is a very delicate task. Besides it is complicated by mercenary interests of projectors and manufacturers. They often try to hide the true causes of wrecks. They do this by all possible means and sometimes they even bribe the members of investigating commissions. These bribes may be very large in order to conceal the true cause of the wreck.

The material of previous sections allow us to indicate characteristic traits of wrecks occurring in systems that are able to loose stability at parameters variations of a certain sign:

1. At a system parameters drift at the moment of a parameter transfer into a dangerous interval in a characteristic polynomial of a system a large positive root appears (see polynomial (30)). This means that a deviation of regulated variables from save values will very quickly increase, wrecks will develop violently like a blow or like a sudden break of a system.

2. If during wreck a dangerous system was not destroyed and was in time switched off by hand or by a system of defense then in some time at a check the system might seem to be quite in order since a parameter whose exit from a safe interval that lead to a wreck (as a result of uncontrolled small variations) this system could again return to an save interval of values by a check moment.

If we take into account all these circumstances a well-known wreck of aerobus A-310 that occurred on the 22d of March, 1994 above Mezdurechensk town was of special interest to us. As a result — all passengers and the crew were killed. The so called "black box" was found. There flight parameters before and after the wreck were registered. The investigation of the "black box" has shown that wreck occurred at the time when the plane was regulated automatically, under the control of an autopilot. Without an apparent cause, quite unexpectedly dangerous deviations of bending and tangage of the plane started quickly to increase from their normal values. So long as the crew tried to transfer to hand control the deviations had increased to such an extent that there was no possibility to return the plane into a normal regime. Aerobus fell and was smashed.

In several months time another aerobus A-310 was flying near Bucharest also under a regime of autopilot. Suddenly (as in the first case) deviations of the tangage and the bend of a plane started increasing from normal values. But here a pilot managed to quickly switch off the autopilot and succeeded in leveling the plane by a hand control. After a successful landing an autopilot and a control system were checked but it turned out that they were in perfect order and worked in a stable regime.

By comparing all these facts a conclusion can be drawn: an autopilot on Aerobuses A-310 have been projected in such a way that it can loose stability at variations of its some parameters or at combinations of variations. And these variations or their combinations became the cause of two stability losses. One of these losses has led to deaths of passengers and the crew. It took place on the 22d of March, 1994 above Mezdurechensk. As we can guess that the calculation of an autopilot and of the whole control system has been made by quick-operating technique calculation. The control system was prepared by means of a mathematical model of a system of equations transformed into a Cauchy form and therefore it was difficult to make apparent its dangerous properties.

The refusal of a control system during the flight near Bucharest has not been investigated but the wreck above Mezdurechensk was investigated in detail by an international commission since Aerobus A-310 was projected and manufactured by a Franco-German firm. But during that fatal flight above Mezdurechensk the Aerobus was under control of the Russian crew. According to the recording of a "black box" it was established that at the moment of wreck in the cabin there were children of the crew's first pilot. Therefore the Russian crew have made a gross blunder against official instructions foreboding the presence of strangers in the cabin of a plane. The commission considered this to be the main cause of the wreck. At the same time it is certain that the main cause of the wreck was the loss of stability by an autopilot and it was necessary to analyze why it was lost. The commission received a report where it was stated that the cause of this wreck may be errors in calculations and projection of control systems in the aerobus. And it was necessary to investigate this version. But no reaction from the commission was received. We can only guess why an international commission did not wish to investigate the causes of a wreck in details. But it is certain that if the cause of a wreck has been proclaimed blunders in projection the responsibility for the wreck would be put to the Franco-German firm. They would have to pay many-million (150 millions of dollars) to families of the perished. But the commission lied the main fault with the Russian crew who had admitted children into the cabin. Therefore the commission was able to trance their fault to the crew.

We see now how difficult it is to reconstruct a true picture in such a difficult affair as wrecks exploration. The following statement is doubtless: since conventional methods of calculating stability and its preservation at parameters variation are based on the investigation of a system characteristic polynomial or a matrix of its coefficients when it is written in a Cauchy form conventional methods can not always (for all systems) give correct results. Differences between transformations that are equivalent in a classical sense and between transformations that are equivalent in a widened sense are not taken into account. Therefore if no additional checks were introduced into the practice of calculation stability investigation (which were described in the above book edited by Leningrad University in 1987 [1]) then wrecks will be inevitable. They will take place if not now then tomorrow. But we must not tolerate such awful wrecks by no means. It is pardonable when wrecks occur as a result of ignorance or as a result of a modern technique imperfection that can not be solved at our time. But it is completely unforgivable when the cause of a wreck is due to laziness, inertness or there is no wish to listen to warnings of scientists who advise to undertake additional checks that have already been proposed and described in scientific literature.

In order to prevent mistakes and wrecks it is necessary to develop a theory of transformations that are equivalent in a widened sense. The whole theory is far from created. In the next section we shall present a method of stability preservation checking and a theory of transformations equivalent in a widened sense for a special case — for differential equations describing non-linear feedback control systems.

§7. Transformations equivalent in a widened sense

In a control theory control objects and control instruments are considered separately as they form controlling interactions on these objects. Equations of control objects are considered given and unchangeable (this simplifies further investigation) and it is admitted that they can be written in a Cauchy form for linear objects with constant coefficients:

$$\dot{x} = Ax + Bu, \tag{40}$$

where x — n-dimensional vector of variables x_1, x_2, …, x_n. Here x_1 are very often deviations from desired values, A — square matrix of coefficients in a dimension $n \times n$, u — controlling interaction (scalar), B — matrix-column of coefficients in a controlling interaction. We are considering a case of one controlling interaction but in a general case there can be several controlling interactions. And we shall not consider a general case. We shall not consider an influence of perturbing interactions as well.

A controlling interaction u is formed as a function of variables x_1, x_2, …, x_n according to the equation

$$W_{n+1}(D)u = W_1(D)x_1 + W_2(D)x_2 + \ldots + W_n(D)x_n, \tag{41}$$

where $W_i(D)$ — polynomials of differentiation operator $D = \frac{d}{dt}$. They must be chosen in such a way that processes in a system (40)–(41) proceeded in a most desired form.

There is a well-developed theory of a choice and calculation of such polynomials. This theory is a section of a general theory of optimal control. Equation (41) is called **a law of formation of a controlling interaction in a circuit of an inverse connection.** [1]

Instead of polynomials $W_i(D)$ in equation (41) only constant coefficients of intensity are often used. And a law of forming an inverse connection takes a more simple form:

$$u = k_1x_1 + k_2x_2 + \ldots + k_nx_n \tag{42}$$

or in a matrix form:

$$u = Kx \tag{43}$$

where K — matrix-line of constant coefficients without zero elements.

If we put (43) into (40) we see that the change of variables x_i in the system (40)–(43) will be described by equations system

$$\dot{x} = (A + BK)x, \tag{44}$$

Since the choice of a matrix-line K is at our disposal we can choose it in such a way that the processes in a control system took place in the most desirable way for us. A characteristic polynomial of a closed system is equal to a determinant of a matrix $\lambda E - A - BK$, where E — singular matrix. If this polynomial is a Hurvitz one than the stability of all solutions x_1, x_2, …, x_n of a control system (40)–(43) is guaranteed. By a proper choice of a matrix-line K stability and some other quality indicators can be as well secured.

In practice very often not all variables x_i can be directly used for the formation of a controlling interaction. Therefore a vector of these variables y_1, y_2, …, y_m that can really be measured at the exit of a control object and apply for the formation of a controlling interaction is introduced. As a rule the dimension of a vector y is less than the dimension of a vector x, $m < n$.

Only in rare cases $m = n$. The connection between x and y is determined by an equation

$$y = Hx, \tag{45}$$

where H — square matrix of a dimension $m \times n$, where $m < n$.

In calculation practice of control systems transformations connected with the change of matrix H are widely used. This is due to different causes. As it was already shown some of regulated variables can be measured with difficulty. So for example in control systems the flight of an airplane such a variable as an angle of attack is of great importance. It enters into equations of a plane as a control object (into equations (40)). At the same time there is no sufficiently single and convenient instruments that could measure this variable. Its value can be determined indirectly — by measuring other variables (an angle of tangage and its derivatives).Therefore such a variable as an angle of attack can not be used directly in channel of an inverse connection.

Besides in order to reduce a number of measuring instruments a series of variables in a channel of inverse connection is declined quite often. These variables are substituted by combinations of other variables and their derivatives that are equal to them on the basis of equations (40).

For example let us consider a control system of orientation in a stabilized platform and let us limit ourselves by a control in one plane. Then equations of a control object will assume a very simple form:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \end{aligned}\right\} \tag{46}$$

where x_1 — angle of platform deviation from a desire position, x_2 — velocity of this deviation, u — moment created by executing engine and playing a role of a control.

A law of forming a control can be selected in the form:

$$u = -(k_1 x_1 + k_2 x_2). \tag{47}$$

An equation (47) is called an equation of a regulator.

If we introduce (47) into (46) and if we exclude a variable x_2 we shall obtain (in relation to x_1) an equation:

$$(D^2 + k_2 D + k_1)x_1 = 0 \tag{48}$$

whose solution is of a form:

$$x_1 = e^{-0{,}5k_2 t}\left(C_1 \sin\sqrt{k_1 - \frac{k_2^2}{4}}t + C_2 \cos\sqrt{k_1 - \frac{k_2^2}{4}}t\right). \tag{49}$$

It is seen that when $k_1 > 0$ and $k_2 > 0$ a control system is stable. And by changing the value of coefficients k_1 and k_2 processes taking place in a control systems can obtain some or other properties. For example if $k_1 > \frac{k_2^2}{4}$ then a function $x_1(t)$ is a damping harmonic vibration but if $k_1 < \frac{k_2^2}{4}$ there will already be no vibrations.

The realization of a control law (47) requires two instruments: one of then measures a deviation angle x_1, another — velocity of its change x_2. One instrument may be enough if the first of equations (46) is used and if we put instead of x_2 a value that is equal to it $\dot{x}_1$ into equation (47). Instead of equation (47) we obtain an equation

$$u = -(k_1 + k_2 D)x_1 \tag{50}$$

If we introduce (50) into (46) we shall again obtain equation (48) with the same solution (49). It must be so as the introduction into equation (47) instead of x_2 a value equal to it x_1 is by all means an equivalent transformation in a classical sense. Note that in this case this transformation is equivalent in a widened sense as well: it is not difficult to check that as a system of equations (45)–(47) so a system (46)–(50) preserve stability at variations of any its coefficients. The substitution of a variable x_2 by a variable that is equal to it x_1 does not change neither a solution (49) or its surrounding. Solutions of equations that are in the surrounding of an equation (48) differ little from solutions (49). But in other cases the change of variables can lead to substantial changes of surrounding in a system.

If we change a regulator (47) by a regulator (50) we have changed a matrix H entering into an equation (45) is used. If a regulator (47) then a matrix H has a form:

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{51}$$

i.e. is a singular matrix and if a regulator (50) is used then it is a matrix-line:

$$H_2 = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{52}$$

But a characteristic polynomial of equations system (40)–(43) does not depend on matrix H. A characteristic polynomial together with all solutions of equations system depend on only matrixes A, B and K. Therefore transformations equivalent in a classical sense changing a number of "real exits", i.e. a matrix H were widely used and even now are used in a calculation practice of control systems.

Now let us consider such mathematical surprises (in more details) that occur at these common transformations that are everywhere applied.

Let us consider a control object described by a system of three first order differential equations:

$$\left.\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1 u \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2 u \\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_3 u \end{aligned}\right\} \tag{53}$$

if a law of forming a controlling interaction a regulator is true:

$$u = k_1 x_1 + k_2 x_2 + k_3 x_3. \tag{54}$$

Let coefficients k_1, k_2 and k_3 be selected in such a way that processes in a control system take place in a desirable for us way. Let a control system be stable and let it preserve stability at parameters variations. We can achieve this by selecting k_1, k_2 and k_3.

Now let us suppose that we have decided to reduce the number of measuring instruments. We have made up our minds not to use variables x_2 and x_3 in a channel of inverse connection but we have changed them by a variable x_1 and its derivatives on the basis of equations (53).

For this it is sufficient to exclude variables x_2 and x_3 from equations (53) and (54) by means of equivalent transformations. Such an exclusion is a difficult task requiring much tolerance. But if you tolerably and carefully carry out all necessary calculations then we shall come to two equations (in relation to two variables x_1 and u that are remaining):

$$\left(m_1D^3 + m_2D^2 + m_3D + m_4\right) x_1 = \left(m_5D^2 + m_6D + m_7\right) u \qquad (55)$$

$$\left(m_8D + m_9\right) u = \left(m_{10}D^2 + m_{11}D + m_{12}\right) x_1 \qquad (56)$$

in which each of coefficients m_1, m_2, …, m_{12} depends on coefficients a_{11}, a_{12}, …, a_{33}; b_1, b_2, b_3; k_1, k_2, k_3 of an initial system (53)–(54). So for example a coefficient m_8 is equal to:

$$m_8 = \frac{b_3(a_{31}k_2 - a_{32}k_1)}{a_{31}(a_{12}a_{31} + a_{22}a_{32}) - a_{32}(a_{11}a_{31} + a_{22}a_{32})}, \qquad (57)$$

and a coefficient m_{10} will be equal to $\frac{m_8}{b_3}$.

A characteristic polynomial of a system of equations (55)–(56) will be a fourth degree polynomial in a general case:

$$n_4\lambda^4 + n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0 \qquad (58)$$

whose coefficients depend on coefficients m_1, m_2, …, m_{12} of a system (55)–(56) and therefore on coefficients a_{11}, …, a_{33}, b_1, b_2, b_3 of an initial system (53)–(54) as well.

So for example a coefficient n_4 is equal to: $n_4 = m_1m_8 - m_5m_{10}$. If a coefficient n_4 in the polynomial (58) is expressed by coefficients of an initial system (53)–(54) it is easy to see that if we have any coefficients in an initial system a coefficient in a largest degree polynomial (58) will always turn into zero. A relation $m_1m_8 - m_5m_{10} = 0$ is always fulfilled.

Thus a polynomial (58) in fact turns into a third degree polynomial and it completely coincides with a characteristic polynomial of an initial system

(53)–(54) which can be obtained without excluding variables x_2 and x_3 directly with the help of linear algebra formulas.

It must be so since the exclusion of variables made without mistakes is an equivalent (in a classical sense) transformation and a characteristic polynomial at such transformations does not change.

But a coefficient in the eldest-fourth degrees in polynomial (50) turns into zero only when all coefficients in a system of equations (55)–(56) and therefore all coefficients to an initial system (53)–(54) are exactly equal to their nominal values. If coefficients of a system (55)–(56) or of an initial system (53)–(54) deviate from their nominal values in the course of exploitation even by infinitely small degree of values then there could not already be an equality $n_4 = 0$ in a polynomial (58). This polynomial will become a fourth degree polynomial and a sign to the eldest coefficient n_4 will depend on unpredicted parameters variations. If a sign in a polynomial (58) of the eldest coefficient will be reverse to the sign of other coefficients then a polynomial (58) will no longer be Hurvitz and a system (55)–(56) will loose stability.

Thus we have come to an important conclusion: an exclusion of variables x_2 and x_3 from system (53)–(54) if it was done correctly and without mistakes is a transformation that is equivalent in a classical sense but not in a widened sense.

System of equations (55)–(56) that appeared as a result of excluding x_2 and x_3 from system (53)–(54) is equivalent to system (53)–(54) in a classical sense and is not equivalent in a widened sense. Solutions of these systems coincide but environments of solutions are different.

If we repeat a similar investigation for control objects of an arbitrary degree of a form (40) when a law of forming a controlling interaction (43) is satisfied we shall come to the following result:

— if two or more variables do not take part during formation of a controlling interaction and they are substituted by combinations of other variables and their derivatives that are equal to them then such a transformation will be equivalent in a classical but not in a widened sense. If one of variables x_1 does not take part in forming a controlling interaction and it is changed by a combination of other variables (equal to it) and their derivatives then such a transformation will be equivalent as well in a classical sense as in a widened sense. The proof of this phenomenon was given in [1] (a number in square columns means a number in a literature list at the end of the book).

Now an example with a system of equations that was considered, earlier

— (25)–(26) and (33)–(35) became more clear. Now we can say that such systems are not rare. There is plenty of them and they can be quite easily encountered: it is sufficient to exclude not less than two variables in any object of control of the type (40) that is not less than of the third degree with a regulator (43) by means of transformations that are equivalent in a classical sense. We shall come to a system which will be equivalent to an initial one in a classical sense but not in a widened sense. There are many such systems and an encounter with such a system can lead to wrecks if no additional calculation is undertaken but restrict ourselves by conventional check of a characteristic polynomial or coefficients matrix written in the Cauchy form.

Here is an example with a fourth degree control object that had been already given in [2].

A control object is described by a system of equations with the following constant coefficients:

$$\left.\begin{aligned} \dot{x}_1 &= -2x_1 - x_2 - 3x_3 - 2x_4 + u \\ \dot{x}_2 &= -x_1 - x_2 - x_3 - x_4 + 2u \\ \dot{x}_3 &= -x_1 - 2x_2 - x_3 - x_4 + u \\ \dot{x}_4 &= -3x_1 - x_2 + x_3 - x_4 + 2u \end{aligned}\right\} \tag{59}$$

and a controlling interaction is formed by according to the law:

$$u = -x_1 - 2x_2 - 3x_3 - x_4 \tag{60}$$

A characteristic polynomial in a system of equations (59)–(60) is of the form:

$$\lambda^4 + 15\lambda^3 + 6\lambda^2 + 2\lambda + 36 \tag{61}$$

and it is of a Hurvitz type. A system (59)–(60) is stable. And it can be proved that this system preserves stability at variations of its coefficients.

If only variables x_1 and x_2 takes part in the formation of a controlling interaction then variables x_3 and x_4 must be excluded from equations (59)–(60) by using only equivalent (in a classical sense) transformations. After undertaking these transformations let us reduce equations (59)–(60) to the form:

$$\left.\begin{aligned} (D^2 + D + 2)x_1 - (3D^2 + 7D + 2)x_2 &= -(5D + 6)u \\ (D^2 + 2)x_1 - (2D^2 + 2)x_2 &= -(3D - 1)u \end{aligned}\right\} \tag{62}$$

$$(7D + 10)u = -(D^2 + 2)x_1 + (4D^2 + 11D)x_2. \tag{63}$$

A characteristic polynomial in a system of equations (62)–(63) (as it can be proved) preserves a form (61) and remains Hurvitz. This once more confirms the following fact: systems of equations (59)–(60) and (62)–(63) are equivalent in a classical sense. But if (for example) in the first of equations (62) a coefficient in D^2x_1 adopts value $1+\varepsilon$ instead of 1 because of an inevitable small drift of parameters in control object then a characteristic polynomial in the system (62)–(63) will become equal to:

$$-2\varepsilon\lambda^5 + (1-3\varepsilon)\lambda^4 + 15\lambda^3 + 6\lambda^2 + 2\lambda + 36 \qquad (64)$$

and when $\varepsilon > 0$ is infinitely small the system becomes unstable. Systems (59)–(60) and (62)–(63) are examples of systems equivalent in a classical sense but not equivalent in a widened sense.

Now let us ask: how will a real control system behave itself if its mathematical models are equivalent (in a classical sense) to systems of equations (59)–(60) and (62)–(63)? Here everything depends on how a controlling interaction is in fact forming. If it is forming from variables x_1, x_2, x_3, x_4 according to a law (60) then a real system of control will preserve stability at variations of its any parameters. If a controlling interaction is formed only from variables x_1 and x_2 and their derivatives according to a law (63) equivalent in a classical sense to (60) then stability of a real system can be lost at infinitely small degree of variations of some parameters or coefficients. But this dangerous property will not be clearly seen if a mathematical model of a system is reduced (as a rule) to a Cauchy form. From this the possibility wrecks appears. Examples of systems similar to systems (59)–(60) and (62)–(63) were given in [1] and [3].

Now let us consider the most general case when in a control object (40) and a controlling interaction is formed according to the law (41) and a connection between variables x_1, x_2, ..., x_n and real exits of a control object is described by an equation (45). Here a criterion can be formulated whose fulfillment guarantees stability preservation at parameters variations. This criterion was developed by the author together with M.A. Galaktionov in a book, at its end [1].

It can be called Yu.P. Petrov and M.A. Galaktionov criterion or "P-G criterion" The criterion is based on the material described in [1]. It concerns with the optimization of control systems by square and root — mean-square value criteria. Here we shall not deal with this criterion. It must be only noted that the criterion consists in the checking that determinants of a series of matrixes are equal to zero. In them matrix H from equation (45) as well enters.

The following criteria can be formulated that allow to differentiate transformations equivalent in a widened sense for linear control systems. For a general case of control objects (40) with inverse connection (41) and a connection matrix H between variables $x_1, \ldots, x_n$ and real exits the following statement can be formulated. It is necessary that transformations that are equivalent in a classical sense be also equivalent in a widened senses as well.

Here they are:

1. It is sufficient that at transformations matrix H from equation (45) does not change.

2. It is necessary and sufficient that transformed equations system satisfy P-G criterion from the book [1].

If an inverse connection is formed according to a law (41) and by means of transformations equivalent in a classical sense p variables x_i are excluded and they are changed by combinations of remained variables and their derivatives (equal to them) then such a transformation is equivalent in a widened sense if $p < 2$ and it will not be transformation equivalent in a widened sense when $p \geq 2$.

Note that these results are obtained only for systems of linear differential equations with constant coefficients for a special case — for systems of equations describing control systems. Such systems consist of equations for control objects (40) and equations for a regulator (41). For a general case systems of several differential equations with constant coefficients (of different degrees) a problem of differentiation between transformations equivalent and not equivalent in a widened sense has not as yet been solved. It is an interesting and important theme for scientific investigations.

§8. Prevention of wrecks and catastrophes

In §6 we discussed wrecks whose causes were drawbacks of conventional projection and calculation methods of control systems. Besides there are differences between transformations equivalent in a classical sense and transformations equivalent in a widened sense and this fact was not taken into account as well. If this difference is perceived it is not at all difficult to prevent wrecks.

For example a characteristic polynomial of a control system can be calculated by means of equations written in relation to such variables in whose function a controlling interaction (of variables y_1, y_2, ..., y_n from equation (45)) is really formed. After we have calculated a polynomial and when we became sure that it was Hurvitz it is necessary to check out additionally:

1. That not a single root of a characteristic polynomial lie on a complex space near an imaginary axis. This is a usual check and it is recommended in text-books on automatic control and it is applied in all projection-construction organizations. Therefore it is necessary to speak about it in a more detail. But the following two checks are not mentioned in text-books although they are also important and necessary.

2. It is necessary to check that a degree of a characteristic polynomial be not less than a sum of degrees of differential equations entering into system.

3. It is necessary to check that a degree of a characteristic polynomial coefficients be not by far (by two-three orders) less than others.

The importance of point 2 can be seen from examples of systems (25)–(26) and (55)–(56). If a characteristic polynomial degree is less than a sum of degrees of differential equations entering into a system it means that the eldest coefficient turns out to be a difference between two similar numbers and due to this cause it turned into a zero. So in a system of equations (55)–(56) the eldest coefficient of a characteristic polynomial (a coefficient in λ^4) turned into zero because it is equal to $m_1 m_8$ - $m_5 m_{10}$. But in system (55)–(56) there is an equality $m_1 m_8 = m_5 m_{10}$. But such an equality can exist only if parameters of a real made "from metal" control system are exactly equal to parameters of a mathematical model. At parameters variations an exact equality can be broken and a member $n_4 \lambda^4$ of the fourth degree with a small coefficient n_4 in a characteristic polynomial appears. If $n_4 > 0$ then a system will be stable. But during exploitation at small variations of coefficients m_1, m_8, m_5 and m_{10} difference $m_1 m_8$ - $m_5 m_{10}$ can change the sign and stability disappears.

The importance of the point 3 can be seen on a system (25)–(26) that we have examined. If variations of coefficients are such that a characteristic polynomial remains a Hurvitz one and it is of fourth degree but with a small coefficient in to the eldest member then this means that this coefficient can change its sign during the variations. The change of a sign will means the loss of stability of a real system.

We see that preservation of degree during transformations of differential equations does not by all means guarantee equivalence in a widened sense. It is also necessary to check a case when the degree does not change but some coefficients become small in comparison with others.

The proposed approach has the following drawback: during calculations by real exits without being reduced to the Cauchy form a normal compiling of programs that will secure calculations can not be used.

Therefore another approach can be possible: a mathematical model of control object is reduced to Cauchy form, a matrix H is written from equation (45). Then after a calculation an additional following check takes place: whether conditions of Petrov-Galaktionov criterion are fulfilled (P-G criteria) that were given in [1].

Such a check is quite reliable and it guarantees from wrecks due to the loss of stability. But in order that the application of the check be convenient we must surely develop the compiling of programs that would secure the check of P-G criteria from [1] since its check "by hand" is in majority of cases not real.

Note that if conditions of P-G criterion are not fulfilled in [1] it is shown what changes must be introduced into a projected control system in order that this system preserve stability at parameters variations.

We see that if the difference between transformations that are equivalent in a classical sense and in a widened sense is perceived, if the danger of wrecks taking place as a result of mixing these conceptions is clear then we can escape wrecks and catastrophes. We can avoid them even by rather simple means.

Here is the main danger: in the majority of projection-construction organizations the necessity of upgrade of conventional methods of calculating and projection up to now has not been perceived. Although the new methods were discussed several times at scientific seminars and although theoretic foundations of additional check methods that help to prevent from wrecks were published in a known book [1] and in scientific and technical magazines

[3, 4, 5] warnings on the incompleteness of conventional calculation and projection methods were published as well all the same advanced check methods did not become everyday practice of construction-projection organizations. Therefore the possibility of inexcusable wrecks is not destroyed, it exists up to now.

It would be quite annoying if advanced calculation methods be put into practice not in our country but by a foreign firm for the first time. In principle it can happen easily as everything was published in Russian magazines. Additional work on compiling programs that will secure calculations is not very large.

At the same time the firm that first of all would put into practice advanced calculation methods would gain great advantage over competitors. Its production, its systems and equipment can be proclaimed to be more secure than the production of firms that do not apply this methods but a conventional one. While struggling for markets the firm that applies advanced calculation methods has serious advantages over other firms.

§9. Non-linear systems. Does the existence of Lyapunov function guarantee the stability preservation at parameters variations?

On the solutions stability of linear differential equations systems with constant coefficients we judge by roots of characteristic polynomial. For systems of non-linear equations one of the most wide-spread and strong methods of checking stability is the method of construction and investigation of Lyapunov functions. These functions were called Lyapunov functions in honor of a great Russian mathematician A.M. Luapunov (1856 — 1918) who in 1892 developed new methods of stability checking.

Let us describe (in short) the main properties of Lyapunov function. Let a system of non-linear independent equations in the Cauchy form be given:

$$\left.\begin{aligned} \tfrac{dx_1}{dt} &= f_1(x_1, \ldots, x_n) \\ &\ldots \\ \tfrac{dx_n}{dt} &= f_n(x_1, \ldots, x_n) \end{aligned}\right\} \tag{65}$$

where $f_i(x_1, \ldots, x_n)$ — non-linear functions (in a general case) of variables $x_1, \ldots, x_n$ and let a zero solution satisfy a system (65). Will this solution be stable? For definition let us mean an asymptotic stability by stability when solutions satisfying initial conditions if $t = 1$ that little differ from zero.

Now let us introduce a function V of variables $x_1, \ldots, x_n$ that is equal to zero only when all variables are equal to zero and is positive at all other combinations of variables values.

The following function can serve as an example of such function:

$$V = x_1^2 + \ldots + x_n^2 \tag{66}$$

Now let us calculate a total derivative of time function V from system (65) solutions. Such a derivative (according to the stability theory terminology) is called a derivative of a function V from a system (65). Let us use a known formula for a total derivative for calculations:

$$\frac{dV}{dt} = \frac{dV}{dx_1} \cdot \frac{dx_1}{dt} + \ldots + \frac{dV}{dx_n} \cdot \frac{dx_n}{dt} \tag{67}$$

and instead of derivatives $\frac{dx_i}{dt}$ let us put their values from equation (65). We shall obtain for the derivative from a system (65) a formula:

$$\frac{dV}{dt} = \frac{dV}{dx_1} \cdot f_1(x_1, \ldots, x_n) + \ldots + \frac{dV}{dx_n} \cdot f_n(x_1, \ldots, x_n). \tag{68}$$

Now let a function V be such that a derivative (68) for all $x_i \neq 0$ be negative. Let us call such function Lyapunov function. In 1892 A.M. Lyapunov proved that if such a function exists then a zero solution of a system (65) is asymptotically stable.

Lyapunov's proof shows that if solutions of a system (65) a derivative of a function V is negative then function V will only decrease in the course of time tending to its least value equal to zero. And since this least value is achieved if $x_1 = x_2 = \ldots = x_n = 0$ then all variables x_i will tend to zero.

Let us consider the following system as an example:

$$\left.\begin{array}{l} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 \end{array}\right\} \tag{69}$$

whose zero solution is by all means stable since all solutions of a system (69) are of the form: $x_1 = c_1 e^{-t}$; $x_2 = c_2 e^{-t}$.

The following function can be taken as Lyapunov function:

$$V = \frac{1}{2}(x_1^2 + x_2^2) \tag{70}$$

(it's a quadratic form of variables x_1 and x_2). Its derivative $\dot{V} = \dot{x}_1 \cdot x_1 + \dot{x}_2 \cdot x_2$ from a system (69) turns out into:

$$\dot{V} = -x_1^2 - x_2^2 \tag{71}$$

and for any values of variables except $x_1 = 1$, $x_2 = 0$ it is negative. Hence it follows that a zero solution of equation (69) is stable. If we could directly find a general solution then we could have made a conclusion about stability on the basis of investigating Lyapunov function (70).

We have enumerated the simplest part of Lyapunov's theory. There exist such Lyapunov functions whose derivatives satisfy (for the system) a more weak condition. They must not be negative but they are only not positive. These and more elegant conditions are considered in a lot of works dedicated to stability theory [6, 7, 8, 9, 10]. The main point is: if one or other Lyapunov function is found then a question about the stability of a zero solution in a non-linear system is solved. But the stability of any solution of a system can be reduced to the investigation of stability of a zero solution. Therefore although it is very difficult to find Lyapunov function as there is no general methods of its finding hundred of books and articles by great many scientists are devoted to the searching of Lyapunov functions, to the stability theory development based on the investigation of these functions. For example in a small book by Ye.A. Barbashin [9] there are references on 190 works by 128 authors devoted to these functions.

But we see from the above statements that in order to judge on the real behavior of an examined system we must have the possibility to judge not only on the solution stability but on stability preservation at inevitable in practice parameters variations as well. Therefore let us pose the following important questions: does the existence of Lyapunov function guarantee the stability of a zero solution if parameters variations are infinitely small?

To our regret the answer to this question can be only negative. Really in order to find Lyapunov function examined equations system is usually reduced to Cauchy form (65). As we have already said that in dependence on the type of initial equations systems that directly issue from the laws of physics or theoretical mechanics of a system equations a transformation to the Cauchy form can not become a transformation that is equivalent in a widened sense (even if it is equivalent in a classical sense).

Let's explain this on an example of equations system (25)–(26) that we had already considered. After it was reduced to a Cauchy form this system turns into equations (33)–(35). If we introduce (35) into (33) we obtain the following system of equations

$$\left.\begin{aligned} \dot{x}_1 &= -3x_1 - x_3 - x_4 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_3 - 2x_4 \end{aligned}\right\} \tag{72}$$

A characteristic polynomial of this system is of the form (28) and its zero solution is apriori stable. Besides the following general theorem was proved about a stable linear equation: such system always have a Lyapunov function in quadratic form. In case of a system (72) it will be a quadratic form of variables x_1, x_3, x_4. But as we saw in §4 an initial system (25)–(26) can loose stability even if its coefficients variations are infinitely small. Therefore even for linear systems the existence of Lyapunov functions does not guarantee stability even if variations are infinitely small. The more so there is no possibility of guaranteeing stability preservations at parameters variations for non-linear systems. The difference is only in the behavior of systems after stability loss: for linear systems stability loss means unlimited increase of variables in the course of time. In non-linear systems (and as it is known non-linear systems more completely describe the behavior of real physical objects) no unlimited increase of variables will be after the loss of stability in the course of time. But in non-linear systems the loss of stability (as a rule) accompanies an exit of variables that are above admissible limits and a system that has lost stability becomes inefficient and can be the cause of hard wreck.

Conventional methods of checking stability based (in particular) on the construction of Lyapunov functions can as well give incorrect answer to the question of stability preservation at inevitable in practice parameters variations. Therefore they can be causes for dangerous wrecks.

In order to prevent errors and wrecks it is necessary to strengthen conventional methods of examining stability by additional checks. We spoke about them in previous sections.

§10. Definitions and theorems

In previous sections we used a free style of stating the material and did not give definitions and theorems. The style of stating the material is not a principal question. While choosing the style of stating the material tastes and liking of readers are of importance. The author must try to make his statements to be easily understood and in the most part admissible for the readers. Recall that such an outstanding scientist as academician Aleksei Nikolayevich Kryilov (1863–1945) did not generally use a word "theorem" in his works. But this did not prevent him to be a universally recognized as a classics of applied mathematics. For the convenience of those readers who are used to other style lets try to state results of previous sections in the language of definitions and theorems. The author asks pardon if such an attempt is not successful.

Let us consider a system of linear equations with constant coefficients and let us number all coefficients entering into the system from m_1 to m_k. A system of equations (55)–(56) with twelve numbered coefficients can serve as an example of such a system.

Definition 1. Let us call a set of differential equations system of the similar structure whose coefficients denoted by $\bar{m}_1$ are subjected to the following inequalities be called ε-environment of an examined system:

$$m_i(1-\varepsilon_i) \leq \bar{m}_i \leq m_i(1+\varepsilon_i), \tag{73}$$

where ε_i — numbers that are small in comparison to 1.

Theorem 1. If an examined system is stable and variations of its coefficients satisfy inequalities (73) then in order that the system preserves stability it is necessary and sufficient that in its ε-environment only stable systems enter.

Proof. If in the ε-environment there is only one unstable system then at coefficients variations of the examined system they can coincide with coefficients of just this unstable system. But this means that an initial system after coefficients variations will loose stability. This proves the necessity of a condition in the theorem. Sufficiency of this condition is evident: if in ε-environment there is not a single unstable system then at any variations satisfying the inequality (73) this system will preserve stability.

This theorem gives a more exact formulation to the statement of §4. From this theorem directly follow consequences that we have already considered:

since the property of stability preservation depends on not the system itself but on its ε-environment then this property can appear and disappear during equivalent (in a classical sense) transformations that do not change the stability of an examined system. Hence follows the necessity of introducing a new mathematical conception: an equivalence in a widened sense. We spoke about it in earlier sections.

For non-linear systems we must introduce a small precision in the definition of ε-environment and in theorem 1. Let us limit ourselves to considering independent systems, i.e. systems in which time t does not evidently enter.

For an independent system of non-linear differential equations with constant coefficients let us enumerate all coefficients. The following system can serve as an example:

$$\left.\begin{aligned} \dot{x}_1 &= m_1 x_2 \\ \dot{x}_2 &= m_2 x_1 + m_3 x_2 + m_4 x_2^3 \end{aligned}\right\} \tag{74}$$

Let us suppose that this system has a stable zero solution. Later preservation of stability of this solution at coefficients variations will be examined.

Definition 2. We shall call a set of differential equations systems of the same structure having a zero solution ε-environment of an envisaged system. Coefficients of each system denoted by $\bar{m}_i$ satisfy inequalities (73).

Theorem 2. In order to preserve stability of a zero solution at coefficients variations it is necessary and sufficient that in ε-environment of an examined system only systems whose zero solutions are stable be found.

The proof of theorem 2 is quite analogous to theorem 1 proof.

§11. A problem of stability preservation

The provision of stability preservation at parameters variations is closely connected with the theory of optimal control.

In the sixties a synthesis theory of optimal controlling interactions in feedback channels was already developed and then a synthesis theory of optimal regulators as well. They secured an essential improvement of control quality in comparison to regulators and systems used earlier.

The application of optimal regulators, the substitution of former control systems by them can lead to great economic effect in civil and defense industries. Therefore theory of optimal control was intensively developed in our country and abroad. Several monographs were published. They were devoted to the synthesis of optimal regulators [11, 12, 13, 14]. But during realization of optimal control systems it was at once discovered that in a series of cases they are able to loose stability even at very small deviations of regulator parameters or control object from calculated values. Surely the discovery of such cases each of which at once threatened with wrecks undermined all faith in the theory of optimal control; closed possibility of its practical application. And besides the cause of this stability loss was not clear. For a long period of time the appearance of regulators that loose stability at parameters variations was thought to depend on the drawbacks of synthesis methods of control systems that were used. Therefore the search of such synthesis methods that would guarantee the best possible control quality was continued. And at the same time such methods would guarantee stability preservation at parameters variations that are inevitable in practice.

Only beginning from 1964 and up to 1973 four monographs were published. They were devoted to new synthesis methods of optimal systems. But invariably these new methods occurred to possess the same drawbacks as the old ones they did not guarantee the stability preservation.

The change came in 1973. At the beginning of the year P.V. Nadezhdin published an article [15]. In it the author once more showed that one more algorithm (that was recently proposed) of optimal control systems synthesis did not guarantee stability preservation at parameters variations as well. The author of the article thought that it was algorithm's drawback and he tried to remove it. But in the same year in a monograph [16] was shown that in reality there are no algorithms free from this drawback since a minimum of quality criterion often lies on the stability boundary and not a single algorithm can change this situation. Therefore it is necessary to stop searching non-existent algorithms. And it is necessary to consider stability preservation at parameters variations as an additional requirement for the realization of which it is necessary to pay a sacrifice in a part of quality criterion.

In order to explain this phenomenon let us take an example. In [11, 16] one-connection control systems whose mathematical model is the following differential equation were considered:

$$A(D)x = B(D)u + \varphi(t), \tag{75}$$

where $A(D) = a_n D^n + a_{n-1} D^{n-1} + \ldots + a_0$; $B(D) = b_m D^m + b_{m-1} D^{m-1} + \ldots + b_0$ — polynomials of differentiation operator $D = \frac{d}{dt}$, x — regulated variable, u — controlling interaction, $\varphi(t)$ perturbing interaction.

As a rule a perturbing interaction is a stationary random process that is not fully known to us. Quite often we know only statistical characteristics of this process and in particular we usually know only a spectral power density which is often called a spectrum of the process $\varphi(t)$ (for short). A spectrum of the process is an even function of a variable — ω-frequency. It can be calculated on the basis of working up perturbing interactions observations. Then it is approximated by an even fractional-rational function:

$$S_\varphi = \frac{a_p \omega^{2p} + a_{p-1} \omega^{2p-2} + \ldots + a_0}{b_q \omega^{2q} + b_{q-1} \omega^{2q-2} + \ldots + b_0} \tag{76}$$

Coefficients a_i and b_j in a formula (76) are chosen from the following conditions: function (76) must (as less as possible) differ from experimental spectrum data in the most interesting for us range of frequencies. The reader can find more detailed information on random processes, their spectra, on frequency ranges that are of great importance to the one or other system in the book [1]. In this book algorithms of optimal regulator synthesis are given. Its mathematical model is a differential equation:

$$W_1(D)u = W_2(D)x, \tag{77}$$

where $W_1(D)$ and $W_2(D)$ — polynomials of differentiation operator. This regulator must secure stability of the system (75)–(77) and a minimum of a root-mean-square quality criterion:

$$J = m^2 < x^2 > + < u^2 >, \tag{78}$$

where $< x^2 >$ and $< u^2 >$ are mean squares of variables x and $u(t)$, i.e.

$$< x^2 >= \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T x^2 dt$$

and analogically

$$< u^2 >= \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T u^2 dt$$

and a constant m^2 is Lagrange multiplier. Criterion (76) reflects as requirements to the quality of regulation so limitations on the value of a controlling interaction. More detailed information on quality criterion (78), on the choice of Lagrange multiplier can be found in the above book [1]. In this book rather complex algorithms for the synthesis of as optimal regulator, i.e. strictly speaking — calculations of operator polynomials coefficients to equation (77). For these algorithms a good compilation of programs is formed. It allows to quickly undergo all necessary calculations and obtain a mathematical model of a regulator that later is realized to a really working electro-drive securing the improvement of control quality.

Here is a simple example.

Let us consider a first order control object:

$$4Dx = (D + 1)u + \varphi(t) \tag{79}$$

with a quality criterion $J = 9 < x^2 > + < u^2 >$ and a perturbing interaction $\varphi(t)$ whose spectral density power is well approximated by the formula:

$$S_\varphi = \frac{2}{\pi} \cdot \frac{1}{1 + \omega^2} \tag{80}$$

Having calculated by the algorithm from [1] we shall see that the minimum of quality criterion we have chosen will secure the following regulator:

$$(3D - 5)u = 12(D + 4)x. \tag{81}$$

The value of quality criterion secured by this regulator is equal to 0,4336 and it is the least value from all possible values. Regulator (81) secures stability of solutions to a system of equations (79)–(81) but this stability is broken if some parameters of a control object (i.e. some coefficients of its mathematical model (79) or some of coefficients in a mathematical model of a regulator (81) deviate from calculated values even by infinitely small values.

Surely hence it follows that regulator (81) is quite unsuitable for any practical application. At the same time it is not difficult to check that regulator (81) really secures the minimum of quality criterion that we have chosen. No other regulator of any other structure or of the same structure but with different coefficients with an equal or better value can not secure quality criterion. To our regret this minimum lies on the boundary of stability and there is no calculation algorithm that can change the situation.

A little later in a monograph [17] the following simple criterion was given: the minimum of quality criterion does not lie on stability boundary. The criterion consists in the fulfillment of an inequality:

$$p \geq m + q - 1 \tag{82}$$

Later this inequality received the name of Yu. Petrov's criterion in technical literature. In this inequality m — degree of a polynomial $B(D)$ in the equation of a control object (75), p and q — degrees of a numerator and a denominator in an analytical approximation of a spectral power density (76). The violation of Yu. Petrov's criterion testifies that minimum of quality criterion (78) lies on the boundary of stability and therefore a regulator helping to obtain the minimum to quality criterion (78) can not secure stability preservation at infinitely small parameters variations and in fact it will not be at all workable. So in an example that we are considering with a control object (79) and spectrum of perturbing interactions (80) we have $m = 1$, $p = 0$, $q = 1$ and Yu. Petrov's criterion is not fulfilled and thus a minimum of quality criterion (78) lies on stability boundary. Therefore we must not be surprised that a system of equations (79)–(81) does not preserve stability even at infinitely small parameters variations. In order that stability be preserved it is necessary to sacrifice a part of quality criterion. Inequality (82) prompts us that to obtain this aim an analytical approximation of spectral power density can be changed. It is used during regulator calculation.

Since for a control object (79) a range of low frequencies is of great importance then an approximation (80) can be changed into approximation

$$S_\varphi = \frac{2}{\pi} \cdot \frac{1 + k^2\omega^2}{1 + \omega^2}, \tag{83}$$

that if coefficient k has reasonable values in the most essential for control object (81) frequency ranges from $\omega = O$ up to $\omega = 0,3$ does not greatly differ from approximation (80). The calculation of quality criterion we surely make by an initial approximation (80). The approximation (83) is only used for regulator calculation but not for calculation of quality criterion. The more is a value of a coefficient k in formula (83) the wider is variations range, deviations of control object parameters or a regulator from nominal values that do not break stability. But the more is a sacrifice in quality criterion.

In particular in order to guarantee control object stability not only at small but at large deviations of parameters from calculated values it is sufficient to choose $k = 0,1$. Such calculations are given in the book [1] on page 145–147. Here instead of regulator (81) we obtain the following regulator:

$$(1,9D - 5,3)u = (13,6D + 48)x, \tag{84}$$

but quality criterion instead of having a minimally possible value $0,4336$ becomes equal to $0,4374$ or it increases by $0,88\%$. Such a small sacrifice in the quality criterion during the choice of a new analytical approximation of spectral power density in a perturbing interaction is due to the fact that it must differ from experimental data (as far as possible) beyond frequency ranges that are substantial for some system (in particular for a system (79)–(84) a frequency range equal to $0 \leq \omega \leq 0,3$ is substantial. The details of calculations can be found in the book [1], pp. 145–147).

The publication of a monograph [16] in 1973 has changed all development way of optimal control theory: searches of inexistent algorithm that secures the minimum of quality criterion (78) and at the same time stability preservation at parameters variations were ceased. Instead we have turned to securing stability preservation at parameters variations as an additional requirement to the system. But nowhere can we find an indication that this break in the development of optimal control theory has been proposed and founded in the works [16] and [17] at the Leningrad university. The first method of guaranteeing stability preservation at parameters variations was proposed in [16], a more advanced method was described in [17], pp. 218–226. The silencing of works [16] and [17] accomplished at the Leningrad university by all means damages the University's prestige and the University's authority. But it is not only the University's priority that matters. Due to this silencing another method of securing stability preservation at parameters variations found wide application in calculation practice. This method is based not on the change of analytical approximation, of spectral power density but on the change of quality criterion. Instead of criterion (78) the following criterion is introduced:

$$J =< u^2 > +m^2 < x^2 > +\lambda_1 < \dot{x}^2 > + \ldots + \lambda_k < [x^{(k)}]^2 >, \qquad (85)$$

where only the first and the second members have a clear physical sense and other members are introduced only in order to secure stability preservation at parameters variations. The more are coefficients $\lambda_1, \ldots, \lambda_k$ the wider is the range of possible parameters deviations from nominal values but the more is the sacrifice in quality criterion value (78). The introduction of criterion (85) instead of criterion (78) is in fact equivalent to the change of analytical spectrum approximation and it also allows to secure stability preservation at parameters variations of a control object or a regulator but the sacrifice of quality criterion required for this can be significantly larger. Here is the cause for this: the addition of new members into criterion (78), the transformation of it into criterion (85) is equivalent to such change of analytical spectrum approximation that takes place in an unknown to us frequency range. If this frequency range is of great importance to the system then the sacrifice in quality criterion turns out to be more hard than while using the methods given in [17]. Thus the silencing of St. Petersburg university's priority costs too much to our economy.

But chiefly the aim has been achieved. After the publication of monographs [16] and [17] a way to direct application of optimal regulators in industry and transport has been opened. Now optimal control does not any more frighten projectors by the loss of stability at parameters variation. Economic gain from the transfer to optimal control is sufficiently great and it can be easily estimated.

A lot of data on the practical application of optimal regulators, on the economic effect achieved, on the improvement of stabilization and tracing systems exactness at the transfer to optimal control are published in a monograph [18].

The most interesting results were obtained during examining stability preservation problems in the course of multidimensional control objects optimization whose mathematical model are vector-matrix equations:

$$\dot{x} = Ax + Bu, \tag{86}$$

where x — vector of regulated variables, u — control, scalar, A and B — matrixes. Optimal control theory allows to calculate connections between an optimal control and regulated variables. And for a very important special case of a quadratic quality criterion this connection occurs to be very simple:

$$u = Kx, \tag{87}$$

where K — matrix-line of constant intensity coefficients. Optimal control theory allows us to calculate optimal values of these coefficients (details see in [1], pp. 206—230). But in the majority of applications we must take into account that not all components of the vector x are can be measured and directly used in feedback channel. Therefore it is necessary to apply a square matrix we have already mentioned:

$$y = Hx, \tag{88}$$

that connects a vector of all regulated variables x and a vector of such variables y which can be measured and directly used in a feedback channel. As a rule the dimension of a vector y is less than the dimension of x. After we have obtained an optimal regulator equation (87) it must be transformed to variables y later. Here we surely will use only equivalent transformations.

By analyzing stability preservation of optimal systems at coefficients variations of a mathematical model in a control object (86) or a regulator (87) the following unexpected mathematical phenomenon turns out. We can conclude that stability preservation at parameters variations depends on how many components in a vector y can not be measured. Besides it depends by how many units the dimension of vector y is less than the vector dimension i.e. depends (finally) on matrix H.

The obtained result was unexpected and rather paradoxical firstly since all transformations were equivalent they were considered unable to change any properties of a transformed system. Secondly, a characteristic polynomial of a system of equations (86)–(87) and hence its all solutions do not depend on matrix H but are completely determined by means of matrixes A, B and K. Note that a characteristic polynomial of a system of equations (86)–(87)

is a determinant of a matrix $\lambda E - A - BK$ that does not depend on matrix H.

In [1] on page 230 an explanation of this phenomenon was given. There it was shown that as stability preservation at parameters variations is not a property of a system itself but is a property of its environment then it can change even during equivalent transformations. In order to see that a projected system will not loose stability at parameters variations it is necessary to check the fulfillment of Yu. Petrov-Galaktionov criterion that is given in [1], pp. 212–230. In this book [1] recommendations were given if Yu. Petrov-Galaktionov criterion is not fulfilled — what changes must be introduced into a projected system in order that it preserves stability at parameters variations.

Note that the dependence of stability preservation property at parameters variations on matrix H is not absolutely unexpected. Already in 1961 in his a well-known work an American mathematician R. Kalman showed that on matrix H depends such a system property as observance (in [1] pp. 137–200 the observance was described in a more detail). But for a long period of time it was thought that this dependence relates only to observance but on matrix H stability preservation property of solutions does not depend since solutions themselves really do not depend on matrix H. Investigations made in [1] have shown that as a matter of fact everything is more complex and more unexpected.

Note that in a recent decade great attention was paid to stability preservation problem not only at infinitely small parameter variations but at finite variations of parameters since in order to prevent wrecks caused by technique the solution of this problem is really very important. This problem deserves such an attitude. The start to such investigations was taken in a very interesting work by V.L. Kharitonov [19]. In it the following problem was examined.

Suppose that some system of differential equations with constant coefficients has the following characteristic polynomial:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0, \tag{89}$$

but all coefficients of this polynomial are known to us only with a certain finite error:

$$\bar{a}_i - \varepsilon_i \leq a_i \leq \bar{a}_i + \varepsilon_i. \tag{90}$$

How to check whether a polynomial is Hurvitz one (89) if its coefficients satisfy inequalities (90) ? Up to 1978 we thought that in order to answer this question it was necessary to check signs in roots real parts of a very large set of polynomials since stability can be lost at complex combinations of different coefficients variations. For example, a coefficient a_0 became larger than its nominal value, i.e. it received a positive variation, a coefficient a_1 received a negative variation, coefficient a_2 — a positive one etc. In all it was necessary to check 2^{n+1} polynomials and if n is large this work is extremely difficult. V.L. Kharitonov in his work [19] showed that we can limit ourselves to much less quantities of checks and therefore we greatly simplify all calculations. Kharitonov's work obtained great approbation in our country and abroad and became well-known. The importance of stability preservation check at parameters variations is now well perceived. Tens of work devoted to this problem were published.

But the following unsolved question remains: if we know that a characteristic polynomial (89) remains Hurvitz at different its coefficients variations does this guarantee that initial system of differential equations for which we have calculated this polynomial will preserve stability? Sorry to say the answer to this question can be only negative. Assumptions published in the book [1] already show that any investigation of a characteristic polynomial can not give us a correct answer to the question about stability preservation. The correctness of the answer depends on matrix H on which a characteristic polynomial itself does not depend.

The methods of checking a sign preservation in polynomial (80) real parts bases on the works V.L. Kharitonov is by all means useful. But in order that it guarantees a correct answer to the question about stability preservation at parameters variations this methods must be supplemented. An additional check can include a check by Petrov–Galaktionov criterion or (in a general case) equivalence check in a widened sense of such initial equations system transformations that were undertaken during calculation of its characteristic polynomial.

The statement from the book [1] about the possibility of appearance and loss of stability preservation property in the course of everywhere used and customary transformations of differential equations at first met incredulous attitude. This statement in reality directly follows from earlier investigations of academician V.B. Rumyantzev, of a member of correspondence of the Russian Academy of Sciences V.I. Zubiv, of V.I. Vorotnikov, of V.S. Yermolin and others. Their investigations were devoted to stability examinations in relation to a part of variables. This investigations are of great value as not in all cases stability is important not by all variables at once. For example, the movement of a shell is determined by six variables: the first three of them determine the movement of a gravity center and other three — turnings round a system of axes connected with the center of gravity. If one of variables reflects an angle of turning round a longitudinal axis of a shell then the value of this variable does not influence on the exactness of hitting a target. The movement of a shell in relation to this variable can be unstable as well. Stability by other variables is important.

As an example let us consider a system of three differential equations:

$$\left.\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - 2x_3 \\ \dot{x}_2 &= 4x_1 + x_2 \\ \dot{x}_3 &= 2x_1 + x_2 - x_3 \end{aligned}\right\} \qquad (91)$$

A characteristic polynomial of this system is equal to

$$\lambda^3 + \lambda^2 - \lambda - 1 = (\lambda + 1)^2(\lambda - 1) \qquad (92)$$

and it has both as positive and as negative roots. Therefore all solutions of system (91) can not be stable. Let us examine them by a variable x_1. Let us introduce a new variable $\mu = x_2 - 2x_3$ and transform a system (91) to new variables with the help of equivalent (in a classical sense) transformations. Since $\dot{\mu} = \dot{x}_2 - 2\dot{x}_3$ then if we take into account the third and the second equations from (91) we obtain $\dot{\mu} = 4x_1 + x_2 - (2x_1 + x_2 - x_3) = \mu$. Finally we obtain in relation to variables x_1 and μ equations:

$$\left.\begin{aligned} \dot{x}_1 + x_1 &= \mu \\ \dot{\mu} + \mu &= 0 \end{aligned}\right\} \qquad (93)$$

System (93) has characteristic polynomial $\lambda_2 + 2\lambda + 1$ with roots $\lambda_1 = \lambda_2 = -1$. Therefore solutions $x_1(t)$ and $\mu(t)$ are stable for any initial conditions whereas solutions x_2 and $x_3(t)$ are unstable (as it can be easily verified).

When V.I. Zubov for the first time considered this example this initiative was met unfavorably. In fact system (91) is a connection system. Therefore it seemed evident from the instability of x_2 and x_3 follows the instability of x_1 as well. But in fact it is not so. We can easily see this if we directly integrate system (91) with initial conditions: $x_1(0) = x_{10}$; $x_2(0) = x_{20}$; $x_3(0) = x_{30}$.

We shall obtain:

$$\left.\begin{array}{l} x_1 = x_{10}e^{-t} + (x_{20} - 2x_{30})te^{-t} \\ x_2 = 2(x_{10} + x_{20} - x_{30})e^{t} + 2(2x_{30} - x_{20})te^{-t} + (2x_{30} - x_{20} - 2x_{10})e^{-t} \\ x_3 = (x_{10} + x_{20} - x_{30})e^{t} + (2x_{30} - x_{20})te^{-t} + (2x_{30} - x_{10} - 2x_{20})e^{-t} \\ \mu = x_2 - 2x_3 = (x_{20} - 2x_{30})e^{-t} \end{array}\right\}$$

Now it is evident that at initial conditions any solutions x_1 and μ are asymptotically stable but x_2 and x_3 — instable.

At the same time we can easily check that stability property by variable x_1 disappears at infinitely small variations of some coefficients in system (91), for example, a coefficient in x_2 of the second equation although all roots of a characteristic polynomial in a system (91) lie far from an imaginary axis. This phenomenon is described in a known monograph [10] from where an example with a system (91) was taken. In monograph [10] this phenomenon was explained in the following way: "a property of asymptotic stability in relation to a part of variables possesses intense sensibility to coefficients variations of a linear system." ([10], p. 79). Just in this V.I. Vorotnikov saw the difference of stability by a part of variables and stability by all variables.

Now we see that in fact there is no such difference. Stability by all variables in some systems can also disappear at infinitely small coefficients variations. But it is very difficult to discover and examine such systems.

At the same time (and it is important) results obtained in the problem of stability preservation at parameters variations and published in [1, 2, 3, 4] are a continuation and further development of previous works on this problem [6, 7, 8, 9, 10] and others — and first of all V.I. Zubov's works.

Strictly speaking even in the book [1] it was clearly shown that conventional methods of checking stability preservation at parameters variations based on the investigation of characteristic polynomial property or coefficients matrix (when written in the Cauchy form) apriori can not always give a correct answer for all systems. Therefore in order to avoid wrecks it is necessary to turn to more advanced methods or example — with using Petrov–Galaktionov criterion from [1]. But the publication of the book [1] has not led to changes in calculations practice of stability and its preservations in construction-projection organizations. In 1990 the author sent direct warnings to a series of technical-scientific magazines that a delay in the application of more advanced methods of checking stability preservation will inevitably lead to wrecks that can surely be avoided. One of warnings was published but only in the magazine "Elektromekanika" ("Electromechanics") — see [4]. In other magazines similar warnings were not published because reviewers made objections to the author's ideas. The analysis of opponents and reviewers objections during a series of discussions and scientific debates is rather instructive.

For example, in one of objections the author stated that if during transformations of equations systems such property of a system as stability preservation at parameters variations changed then the applied transformations must be inequivalent and a mistake had stolen in. Such an objection shows that even in the minds of highly qualified specialists there is no unity in specifying the conception of transformations equivalence.

A classical definition of equivalent transformation includes the inalternability of a set of solutions in a transformed system (see "Mathematical Encyclopedia", vol. 4, p. 800, M., Sovetskaya Entziklopediya, 1984). Solutions during equivalent transformations remain unchangeable. But stability preservation property at parameters variations depends not on solutions themselves but on their environment. Therefore it can change (appear or disappear) during equivalent (in a classical sense) transformations of equations.

Against an example with equations (33)–(35) and (25)–(26) if a system of equations preserves stability at sufficiently small variations of any coefficients during equivalent transformations and turns out into a system that is able to loose stability at infinitely small variations the following objection was put. Since during transformations a differentiation operation was used then the following cause of loosing stability can be indicated: during transformations multiplication was undertaken not by an operator of Hurvitz polynomial. The possibility of loosing stability at some combinations of differentiation operations has been known for a long period of time and we examined this possibility in §2 (An example with equation (11). After multiplication by operator polynomial $D - 1$ it turns into an equation (22) with unstable solution). But the mechanism of stability loss here is of quite another kind. The loss of stability is not connected with coefficients variations and does not depend on them. Simply in the solution an exponentially increasing member appears whose exponent index does not depend on parameters variations but is defined only by means of a root of such non-Hurvitz polynomial of differentiation operator by which the equation was multiplied. This cause of possible stability loss has been known for a long period of time. In [1, 2, 4] quite another cause was discovered that was connected with a great difference between transformations equivalent in a classical and widened senses. During the loss of stability due to the above cause in the solutions of linear differential equations systems with constant coefficients an exponentially increasing member appears but its exponent index depends on the value of parameters variations.

Some of those who opposed to the author indicated that coefficients variations in an initial and transformed equations are difficult to compare between themselves since during transformations of equations coefficients also change their value. And in principle there can be cases when to small changes of coefficients in an initial equation large changes in coefficients of a transformed equation will correspond. Such cases are rare but the are possible.

In order to explain the doubts that arise a direct investigation of variations of a real parameter having a clear physical sense in stability was undertaken. A constant current drive working as actuating mechanism of steering was taken. Equation of moments equilibrium by electro-drive force can be written in the form (as it is well-known):

$$m\frac{d\omega}{dt} = i - k\omega - M_c, \tag{95}$$

where ω — frequency of rotation, m — mechanical constant of time, depending on moment of inertia of an electro-drive, its magnetic flux etc., i — armature current that can be regulated and therefore it plays the role of a control, $k\omega$ — resistance moment on the shaft depending on their frequency of rotation, M_c — moment of resistance of an a actuating mechanism. Later we shall denote coefficient k assume to be equal to 2. Then equation (95) can be written as follows:

$$m\dot{x} = -2x_1 + x_2 + x_3. \tag{96}$$

Now let us examine an electro-drive in which a moment of resistance x_3 is connected with the position of a steering x_4 by differential equations:

$$\left.\begin{aligned} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_3 - 2x_4 \end{aligned}\right\} \tag{97}$$

Then a system of equations (96)–(97) will be a mathematical model of a control object. Now let a controlling interaction x_2 be formed according to the equation

$$x_2 = -x_1 - 2x_3 - x_4, \tag{98}$$

that coincides with an equation (35) that is already known to as. Note that equations (33) examined earlier are a special case of equations (96)–(97) if $m = 1$.

The value $m = 1$ will be considered a nominal value of a parameters m and we shall examine the influence of its variations on the stability of solutions. A characteristic polynomial of equations (96), (97), (98) that are a mathematical model of a closed system as can be easily calculated is of the form:

$$m\lambda^3 + (3 + 2m)\lambda^2 + (6 + m)\lambda + 3 = (m\lambda + 3)(\lambda + 1)^2. \tag{99}$$

Its roots are equal to $\lambda_1 = -\frac{3}{m}$; $\lambda_2 = \lambda_3 = -1$. We see that a characteristic polynomial remains Hurvitz for all values of parameter m lying in the limits of $0 < m < \infty$ but this means that solutions of equations system (96), (97), (98) remain stable not only at small but at large deviations of parameter m (a mechanical constant of time of an electro-drive) from its nominal value $m = 1$. Surely we shall come to the same result if we begin to examine stability preservation of solutions in a system of equations (96), (97), (98) with the help of a methods proposed by V.L. Kharitonov [19] and if we use methods of robust control [20] that is so popular in recent years.

Note that the formation of a controlling interaction in the form of (98) — i.e. in the form of a linear function of all coordinates — is widely recommended in control theory beginning from a widely-known monograph by A.M. Lyetov [13]. This method is widely used.

Now let us suppose that a controlling interaction x_2 can be formed only in a function of a variable x_1 and its derivatives. As we have already shown this is a case rather often and as a rule such a situation is connected with the impossibility of measuring in a part of variables and with the convenience of realization etc. Therefore we must exclude variables x_3 and x_4 from equations (96), (97), (98) by means of equivalent (in a classical sense) transformations. Recall that a conception of transformations equivalent in widened sense was introduced by the author only in 1992 in [2]. Therefore all conventional methods of calculations and projection naturally apply transformations that are equivalent in a classical sense. In 1987 it was stated that these transformations can change stability preservation property at parameters variations (see [1]). But at that time it did not attract any attention. If we use transformations that are equivalent in a classical sense after excluding x_3 and x_4 we shall come from equations (96) ,(97), (98) to equations

$$\left.\begin{array}{l}[mD^3+(2+m)D^2+(4+m)D+2]\,x_1=(D+1)^2x_2\\(D+1)x_2=(D^2+4D+5)x_1\end{array}\right\}\qquad(100)$$

A characteristic polynomial of this system of equations (as it can be easily calculated) is equal to

$$(1-m)\lambda^4+(4-3m)\lambda^3+(8-3m)\lambda^2+(8-m)\lambda+3.\qquad(101)$$

When $m=1$ i.e. if a parameter m has a nominal value a polynomial (101) coincides with a polynomial (99). This once more confirms us that systems of equations (96), (97), (98) and (100) are equivalent between themselves in a classical sense if $m=1$.

But polynomial (101) differs from polynomial (98). It stops to be Hurvitz if a parameter m exceeds a nominal value $m=1$ even by infinitely small value.

Therefore the examination of the influence of an actual physical parameter variations — a mechanical constant of time in an electro-drive — on the stability preservation of solutions in differential equations systems describing a closed control system once more assures us that stability preservation property can appear and disappear during equivalent (in a classical sense) transformations of equations. Therefore if we do not take into account a principal difference between these transformations this leads to errors during calculations and projection. And as a result — wreck and catastrophes. While discussing statements and examples given in [1], [4] the following objection was introduced for many times: examined examples are related to well-developed theory of singularly-perturbed differential equations. As it is known such changes of an equation coefficients that lead to the change of its degree are called singular.

This objection is groundless since the theory of singularly-perturbing equations is really developed but it relates to equations having small parameters in the eldest derivatives, to equations of the type:

$$\left.\begin{array}{l}\frac{dy}{dt}=f_y(y;z)\\\varepsilon_z\frac{dz}{dt}=f_z(y;z)\end{array}\right\}\qquad(102)$$

where a total vector of variables y and z of a dimension n is divided into two sub-vectors: a sub-vector y of dimension m where $m<n$ and sub-vector z

of dimension $n - m$ and parameters ε_z are small. In the theory of singularly-perturbing equations solutions change of equations (102) in comparison to such equations (102) in which $\varepsilon = 0$ is investigated.

When $\varepsilon \neq 0$ a degree of equations (102) increases in comparison to the case when $\varepsilon = 0$. It is clear that solutions of higher degree equations can substantially differ (also by stability) from such equations (102) in which $\varepsilon = 0$ (which is beyond doubts). Note that in a singularly-perturbing system (102) small coefficients are apriori present and variations of these small coefficients are small by an absolute value. But no means-in relation to $\varepsilon = 0$. If even $\varepsilon = 0,00001$ then this is conditionally speaking "infinitely more" than $\varepsilon = 0$.

In examples that were examined (and in examples given earlier in [1, 2, 3, 4]) quite another phenomenon was examined: not a single of coefficients in examined equations (25)–(26), (33)–(35), (96), (97), (98) and (100) is small. All coefficients are sufficiently large and are approximately equal to several 1. Small variations (small not only by absolute value but in relation to nominal values) undergo large coefficients. Therefore equations systems that we examine do not relate to singularly-perturbing equations. In systems (considered by us) a new phenomenon was found: the possibility of appearing and disappearing stability preservation property at parameters variations after equivalent (in a classical sense) transformations of equations.

This phenomenon is of great practical value since if we do not take it into account during projection the phenomenon can become a cause of serious wrecks and catastrophes.

We are not at all surprised that this phenomenon (which was published in [1] then in [4, 2, 5] and which was widely discussed beginning from 1990) and practical conclusions (made from it) and not at once acknowledged by scientists. But in 1994 after a discussion that occurred during three years one of the most frequently cited journals "Avtomatika i telemekanika" (automatics and telemechanics) published an article [3] where more advanced methods of stability check were given (in short). There a direct warning was expressed: uncritical use of conventional calculation methods can lead to serious wrecks. After the publication in this journal magazine, main statements introduced by the author can be said to obtain recognition among scientists. But these advanced methods have not as yet found wide application in construction-projection organizations and such attention to the invention is quite unreasonable and disturbing.

Let us consider the matter of things in atomic electric stations. The date of service of nuclear reactors themselves is great and it is much more than the date of service for different auxiliary equipment in the station (pumps, drives and systems of controlling them). Therefore in the course of exploitation instead of equipment that has worked out its term new equipment is mounted, often more advanced one, this is good.

But if this equipment in the course of projection was checked in stability preservation by means of calculating on quick-operating calculation machines but by conventional methods without additional check the necessity of which was stated in [1, 4, 3, 5] then this equipment can turn out to be able to loose stability at an unforeseen time moment and it can create a wrecking situation. Surely systems and aggregates of atomic electric stations are held in store, are equipped by defense means. Therefore not each stability loss will surely lead to dangerous wrecks. It is so. But at the same time each loss of stability, each refusal creates a dangerous situation that can grow into a wreck. And if there is any possibility to prevent such situations then they must be prevented. A game with "atomic fire", light minded attitude to wrecks whose probable sources are named and published is a crime. Thus it is quite evident that advanced methods of stability preservation, additional checking must be at once applied during projection of equipment for atomic stations, the more that the use of advanced calculation method that reduced the possibility of wrecks does not require great expenses. Only a very moderate expense is necessary in order to develop programs compilation that will secure calculations. But methods of additional stability checking have not been used by the majority of projectors and scientists as they were not inclined to realize anything new, unknown. Sometimes during discussions comical and stupid situations occur. At some organization connected with the development of equipment for atomic energetic at a scientific-technical Council a member of the Council said: "For our branch of industry the reduction of wrecks probability is not actual. Only chemists have serious wrecks but in our branch it is out of the question. For example I — myself took part in the abolition of results of a wreck and have obtained 200 "Roentgens" but as you see I am sitting before you safe and sound. Therefore it is not necessary to change anything in our branch of industry, let the chemists do it".

In Russia there is such an organization as Atomnadzor (The State inspection in the atomic industry). To our mind this organization must realize in practice its rights in blocking possible wrecks at atomic electrostations, it must secure the safety of atomic stations. But it seems that from great rights that Gosatomnadzor possesses and their realization lies a gulf. The author spoke with members of this organization — of its north-west branch — in Petersburg and — of its central part — in Moscow. They politely listened to the author's explanations, his method of additional checking stability was discussed, they read conclusions of scientific seminars that recommended to realize these methods in practice — for projection the equipment. The author's inventions was developed at the Petersburg state university in order to prevent wrecks. But in fact "Gosatomnadzor" has done nothing to help and realized these advanced methods into practice of projectors.

After such an attitude we must not be astonished that so much wrecks occurred in Russian industry in general and in atomic energy in particular. The burden of not trying to prevent wrecks is great.

The application of more advanced calculation and projection methods, the reduction of wrecks probability it seems wholly depends on personal competence, on personal conscience. Not all specialists remained indifferent to improving the safety of projecting stable equipment. For example, member of correspondence of the Russian Academy of Sciences Yua.B. Danilyevich took an active part in the works devoted to preventing wrecks in energetic, s-energetics in the development of more advanced calculation methods that secure stability preservation at parameters variations. The chief engineer of one of projecting organizations took an initiative in applying new additional calculation methods of stability checking in his organization. He closed ways to wrecks appearance connected with the incompleteness of conventional calculation and projection methods there. In other organizations this way is not closed. We must work more and more in order to widen the application of advanced scientific methods. The author hopes that the publication of this work will in practice help to apply the above methods in order to preserve stability at parameters variations. By this we shall reduce the probability of wrecks and catastrophes. The time has come when all inspecting organizations and the authorities must adopt effective means for the prevention of such wrecks whose causes are found and conclusions which are made by scientists. When it concerns people's lives inactivity becomes criminal.

§12. Teachers of mathematics are proposed to discuss a new theme at mathematical circles in high schools.

The most exciting themes to be discussed at mathematical circles in high schools are those that are in some way connected with problems examined by modern mathematical science. To our regret it is very difficult to select such themes since problems of modern science are extremely complex and as a rule are quite out of hifh school student understanding.

But one of interesting and up to date theme can be proposed. It is concerned with the problem of correctness change during solution search of algebraic equations by means of equivalent transformations.

High school students are well-acquainted with simple algebraic equations. They can transform them (to transfer members from the left side to the right side with changing a sign; to divide and multiply all members by a number that is not equal to zero; to add one equation to the other etc.) All such transformations (if they are correctly carried out) are equivalent transformations, i.e. all solutions of transformed equations coincide with solutions of initial equations. Rules for the transformation of equations enter into the school curriculum.

During the work at a mathematical circle a conception of solutions correctness must be also given. Since in practical problems all coefficients of equations are almost always known only with limited exactness then a solution of an equation has any practical sense only if it is correctly posed (i.e. when to small coefficients changes correspond small solutions changes). It is not difficult to give examples of correctly and incorrectly posed problems. The most simple method of checking correctness is to repeat the solution for slightly changed coefficients. If the solution greatly changes then the problem is incorrectly posed.

Later it is not difficult to explain that we can boldly use equivalent transformations only because we are sure that equivalent transformations do not change correctness. In the majority of cases it is really so (examples can be easily given). If an initial equation is correct usually correctness is preserved after transformations. It is so almost always. I think that it will be very interesting to find that quite recently new phenomena were discovered: when in special cases equivalent transformation lead to correctness change.

First of all let us consider a system of two linear homogeneous equations with a parameter λ:

$$\begin{cases} (1-\lambda)x_1 + 3x_2 = 0 \\ x_1 + (3-\lambda)x_2 = 0 \end{cases} \tag{104}$$

Since system (104) is homogeneous it naturally has the following zero solutions: $x_1 = 0$; $x_2 = 0$. Now let's put the following problem: to find such values of parameter λ at which system (104) has nonzero solutions.

The teacher can explain that systems of equations that are similar to equations (104) but of higher degree (consisting of four, ten and more equations) often occur in applications. Many important oscillations calculations for different mechanical construction, computating for processes in control systems, electro-drives and even some problems in astronomy and celestial mechanics ("a secular equation") can be reduced to the search of such values of parameter λ at which this systems have nonzero solutions.

In order to solve the above problem we can use the method of variables exclusion during equivalent transformations. For this we must multiply all members of the first equation from (104) by -1 and members of the second one by $(1-\lambda)$. We shall obtain:

$$\begin{cases} -(1-\lambda)x_1 - 3x_2 = 0 \\ (1-\lambda)x_1 + (3-4\lambda+\lambda^2)x_2 = 0 \end{cases} \tag{105}$$

If we put together both equations and exclude x_1 we shall obtain one equation in relation to x_2:

$$(\lambda^2 - 4\lambda)x_2 = 0 \tag{106}$$

Now it is clearly seen that a nonzero solution is possible when $x = 0$ and $\lambda = 4$. By introducing these values into (104) we shall obtain (if $\lambda = 0$)

$$\begin{cases} x_1 + 3x_2 = 0 \\ x_1 + 3x_2 = 0 \end{cases} \tag{107}$$

and if $\lambda = 4$ we have:

$$\begin{cases} -3x_1 + 3x_2 = 0 \\ x_1 - x_2 = 0 \end{cases} \tag{108}$$

It is at once clear that infinitely many other values x_1 and x_2 satisfy equations (107) and (108) except the solutions $x_1 = x_2 = 0$.

Note that when we multiply the second equation from (104) by an expression $(1-\lambda)$ that is equal to zero if $\lambda = 1$ we can obtain a redundant root

$\lambda = 1$. But if we introduce it into (104) we shall obtain a system

$$\begin{cases} 3x_2 = 0 \\ x_1 + 2x_2 = 0 \end{cases} \tag{109}$$

that has no other solutions except $x_1 = x_2 = 0$. Therefore a value $\lambda = 1$ is not a solution. If we take this statement into account equation (106) is equivalent to a system (101). And by examining it we can easily make a conclusion at what values of parameter λ system (107) has nonzero solutions. By means of rather easy (but cumbersome) computations the following can be checked: if we slightly change coefficients of system (104) - for example in the first equation if we put $3.01x_2$ instead of $3x_2$, in the second one - if we put $0.99x_1$ etc. - then values of parameter λ will only slightly differ from those that we have found already: $\lambda_1 = 0$; $\lambda_2 = 4$.

The above problem is correctly posed for a system of equations (104). If we slightly change coefficients (a unit and a four) in equation (106) we again see that values that we have found earlier $x_1 = 0$; $x_2 = 4$ will change little. In this case during equivalent transfer from system (104) to equation (106) correctness is preserved. But correctness can be unpreserved. Really (as we have spoken about earlier) in practice we have to deal with systems like (107) but in them there are much more equations. When the number of equations is four interesting phenomena appear.

Let us consider a system of four linear homogeneous equations with parameter λ:

$$\begin{cases} (2+\lambda)x_1 - x_2 - x_4 = 0 \\ \lambda x_2 - x_3 = 0 \\ x_2 + (2+\lambda)x_3 = 0 \\ x_1 + 2x_2 + x_3 + x_4 = 0 \end{cases} \tag{110}$$

for which let us examine the same problem: at what values of parameter λ nonzero solutions are possible? We shall find a solution by means of variables exclusion. While undertaken equivalent transformations (multiplications and additions) we shall exclude variables x_1 and x_2 we shall obtain the following system of two equations in relation to variables x_3 and x_4:

$$\begin{cases} (\lambda^3 + 4\lambda^2 + 5\lambda + 2)x_3 = (\lambda^2 + 2\lambda + 1)x_4 \\ (\lambda^2 + 4\lambda + 5)x_3 = (\lambda + 1)x_4 \end{cases} \tag{111}$$

If we now exclude a variable x_3 by means of multiplying the first equation from (111) by $\lambda^2 + 4\lambda + 5$ and the second one by $\lambda^3 + 4\lambda^2 + 5\lambda + 2$ and if we put them together then we shall come to third degree equation in relation to x_4:

$$(\lambda^3 + 5\lambda^2 + 7\lambda + 3)x_4 = 0 \qquad (112)$$

It is not difficult to check that a polynomial that is in the bracket turns into zero if $\lambda_1 = \lambda_2 = -1$ (a double root) and if $\lambda_3 = -3$. Therefore a system of four homogeneous linear equations (110) will have nonzero solutions when $\lambda_1 = \lambda_2 = -1$; $\lambda_3 = -3$. If we introduce them into (110) we shall see that it is really so. But an equation that is in relation with x_4 is obtained by means of excluding variable x_3 from system (111). This equation will be of the type (112) only when these coefficients are exactly equal to calculated ones. Let us suppose that in the first equation from (111) a coefficient in λ^2 in a member $\lambda^2 + 2\lambda + 1$ is equal not to 1 but to $1 + \varepsilon$ where ε is a small number. Then equation (112) will become

$$(\varepsilon\lambda^4 + \lambda^3 + 5\lambda^2 + 7\lambda + 3)x_4 = 0, \qquad (113)$$

It is clear that at infinitely small numbers ε of values in λ at which nonzero solutions are possible will already be not three but four values. At small ε the fourth value will be (by its module) very large. It will be much more than other three values. For small ε will be $\lambda_4 = \frac{1}{\varepsilon}$ approximately.

Thus we may conclude that the problem on the determination of such values in parameter λ at which system (111) will have nonzero solutions is an incorrectly posed problem: since an infinitely small change of some coefficients changes its solutions essentially (even the number of solutions changes) although system (111) has been obtained from system (110) by means of equivalent transformations (multiplications and additions). Systems (110) and (111) are equivalent (in a classical sense) between themselves: really if coefficients are exactly known and are integral numbers then they have the same solutions: $x_1 = x_2 = -1$; $x_3 = -3$. System (110) is correctly posed but system (111) - is not. Systems (110) and (111) are not equivalent to each other in a widened sense. A transformation that has transformed system (110) into system (111) is an example of a transformation that is equivalent in a classical sense but not equivalent in a widened sense.

Now let's consider the consequences. While coefficients from (110) are integral numbers there is no trouble. But more often we obtain coefficients during experiments or measurements. Then they are written with a finite number of decimal signs. Therefore rounding off errors are inevitable. For system (111) any, infinitely small inevitable error in coefficients lead to a gross error in results of computations. Instead of three values of parameter λ we obtain four values. And the magnitude of the fourth value of parameter λ depends on the value of errors in coefficients.

As a result of such errors wrecks and catastrophes can occur and often occur. A teacher can give a lot of exciting examples (besides these that have already been given in previous sections). They will increase the high school students' interest in difficult mathematical problems. The search of new examples in incorrect systems and their transformations can become an interesting creative problem such transformations that are equivalent in a classical but not in widened sense and thus change problem correctness are also very exciting. Up to date we know very few such transformations. The search of any new examples is very exciting.

It must be stressed that we are speaking and examining quite a new phenomenon in comparison to all we have known before. It is well-known that a lot of transformations that seem to be (at first sight) equivalent in fact lead to the loss of some root or to the appearance of new ones. For example it takes place when the right and the left sides of an equation are multiplied by an expression that turns to zero at values of variable that do not coincide with roots. But in these cases redundant roots can not naturally depend on coefficients variations in an initial equation. In systems (110) and (111) deal with another phenomenon: both systems are equivalent to each other in a classical sense and if there is no errors in setting coefficients. Then these systems have the same three values of λ at which nonzero solutions are possible. The fourth value of λ that we have in (111) fully depends on variations of some coefficients in equation (111). The difference of this value from other values is great even when variations of coefficients are infinitely small or if errors in computations and their settings are infinitely small.

Also note that in calculation theory there are well-known cases when a small (but finite) errors in some coefficient after a transformation of any kind increases by some times. Surely we must escape such transformations. But a phenomenon that we examine is of a different kind. It is clear that even infinitely small change of some coefficients lead to a gross change of a solution. This means that the change of correctness in our problem has taken place. Thus in fact quite a new phenomenon appears. It is of great practical value for computation and this phenomenon has not been investigated to the end for the present.

§13. A general problem on computations and correctness reliability of mathematical models. The computation of matrix principal values. Adjacent problems.

During practical computations when coefficients and parameters of a mathematical model are almost always known only with limited exactness apriori results of computation can be reliable only for correctly posed problems when at small variations of coefficients in parameters solutions also change little.

There is a well-developed theory on making distinctions between correctly and incorrectly posed problems and correctly and incorrectly posed mathematical models [21, 22]. Note that in recent years incorrectly posed problems also found solutions. But here particular methods are required described in [21, 22]. If we unexpectedly meet an incorrectly posed problem and try to solve it by means of usual methods as a correctly posed problem almost always we come to erroneous result. When in [1-5] the possibility of correctness change of a mathematical model during equivalent (in a classical sense) transformations was found it become evident that it is necessary to check the correctness of a mathematical model itself but it is necessary to check the correctness of transformations that it undergoes.

In §§3-7 examples of equivalent (in a classical sense) transformations that change correctness for a system of linear differential equations. And in §12 the most simple similar example for algebraic equations was considered.

Now let us examine a general problem on the solution of linear algebraic equations and on calculation of matrixes principal values.

We shall examine a system of n homogeneous linear equations with n unknown values:

$$Ax = \lambda x, \tag{114}$$

where x — n - dimensional vector, A — a square coefficient matrix of $m \times n$ dimension, λ — parameter. Equation (114) can be written as well:

$$(A - \lambda E)x = 0, \tag{115}$$

where E — $n \times n$ dimensional singular matrix. Equations (114) and (115) have nonzero solutions only for such values of λ at which matrix $(A - \lambda E)$ determinant turns into zero, i.e.

$$det(A - \lambda E) = 0 \tag{116}$$

If we decompose a determinant by degrees of λ then we obtain a polynomial of n degree:

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_0 \tag{117}$$

that is called a characteristic polynomial of matrix A and its roots are called principal values or characteristic numbers of matrix A.

The problem on the calculation of principal values is of great practical importance since the necessity of their computation occurs in many fields of applications. Principal values are computed during solutions of differential equations systems, during computation of free frequencies of oscillations in mechanical and electrical systems, in astronomy and celestial mechanics during the solution of the so-called "secular equation" and in many other fields.

To problems on the calculation of principal values and on their correctness check an immense list of publications is dedicated [23, 24]. So in the bibliography of the book [24] 217 names of works dedicated to this problem are given.

We shall examine not a problem of matrix principal values in its classical sense but we shall examine an adjacent problem on the computation of parameter λ values at which nonzero solutions of equations system (114) exist but with additional relations that do not contain derivatives between variables.

The following system can serve as an example:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = \lambda x_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = \lambda x_3 \\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = 0 \end{cases} \tag{118}$$

in which the fourth equation does not contain λ. There can be several equations that do not contain λ. In a matrix form these equations can be written as follows:

$$(A - \lambda\bar{E})x = 0, \tag{119}$$

where now (in contrast with equation (115) $\bar{E}$ — not a singular but a quasi-singular matrix, i.e. a matrix in which, firstly, all elements that lie outside the main diagonal are equal to zero and, secondly, r elements on the main diagonal are equal to zero. So for a system of equations (118) matrix $\bar{E}$ has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{120}$$

an element on a diagonal on the last line is equal to zero. Problems that are reduced to the examination of equations (119) can be often encountered in applications. The problem on the computation of small oscillations frequencies in mechanical systems with holonomous connections, holonomous automatic systems [25] and many control systems is of the same kind of problems.

System (119) has nonzero solutions only for such values of λ for which matrix $(A - \lambda \bar{E})$ determinant turns into zero. These values will be roots of a polynomial:

$$det(A - \lambda \bar{E}) \tag{121}$$

whose degree in a general case is equal to $n - r$.

In particular for system (118) polynomial (121) has in a general case a third degree and it has three roots.

Let us solve a system of equations (118) by means of successive exclusion of variables. If we multiply the first equation from (118) by a_{21}, the second one - by $(a_{11} - \lambda)$ and if we add them to each other than we shall obtain an equation that does not contain x_1. Later if we carry out the same operations with the second and the third equation from (118) and with the third and the fourth ones then we shall come to a system of three equations in relation to x_2, x_3 and x_4:

$$\begin{cases} b_{22}^{(2)} x_2 + b_{23}^{(1)} x_3 + b_{24}^{(1)} x_4 = 0 \\ b_{32}^{(1)} x_2 + b_{33}^{(1)} x_3 + b_{34}^{(0)} x_4 = 0 \\ b_{42}^{(0)} x_2 + b_{43}^{(1)} x_3 + b_{44}^{(0)} x_4 = 0 \end{cases} \tag{122}$$

where by $b_{ij}^{(k)}$ we denote not numbers but polynomials including different degrees of parameter λ. The upper index reflects a degree of a polynomial. Equations system (122) is equivalent to system (118).

If we carry out exclusion operation of a variable x_2 in a system of equations (122) we shall came to the following system of two equations in relation to x_3 and x_4:

$$\begin{cases} c_{33}^{(3)} x_3 + c_{34}^{(2)} x_4 = 0 \\ c_{43}^{(2)} x_3 + c_{44}^{(1)} x_4 = 0 \end{cases} \tag{123}$$

Polynomials of λ we denote by $c_{ij}^{(k)}$ and polynomial $c_{33}^{(3)}$ is a polynomial of the third degree, polynomials $c_{34}^{(2)}$ and $c_{43}^{(2)}$ have second degree but $c_{44}^{(1)}$ — the first one, i.e.

$$c_{33}^{(3)} = m_1\lambda^3 + m_2\lambda^2 + m_3\lambda + m_4 \tag{124}$$

$$c_{34}^{(2)} = m_5\lambda^2 + m_6\lambda + m_7 \tag{125}$$

$$c_{43}^{(2)} = m_8\lambda^2 + m_9\lambda + m_{10} \tag{126}$$

$$c_{44}^{(1)} = m_{11}\lambda + m_{12} \tag{127}$$

at the same time coefficients $m_1, m_2, \ldots, m_{12}$ can be expressed by coefficients a_{11}, *dots* , a_{44} of an initial system (118). If we exclude variable x_3 from a system of equations (123) we shall obtain an equation:

$$(c_{34}^{(2)}c_{43}^{(2)} - c_{33}^{(3)}c_{44}^{(1)}) = 0 \tag{128}$$

from which it is clear that values of λ at which an initial system (118) has nonzero solutions are roots of a polynomial:

$$c_{34}^{(2)}c_{43}^{(2)} - c_{33}^{(3)}c_{44}^{(1)}. \tag{129}$$

At first glance it seems that polynomial (129) is a polynomial of the fourth degree but calculations show that when $a_{31} = 0$ or a_{44} the following equality appears:

$$m_5m_8 - m_1m_{11} \tag{130}$$

and a coefficient in λ^4 in this case identically is equal to zero at any values of other coefficients a_{ij} in an initial system of equations (118). As a result polynomial (129) is in general case a polynomial of the third degree and it has three roots that are the sought values of parameter λ at which system (118) has nonzero solutions.

But equality (130) will be fulfilled only when all coefficients m_1; m_5; m_3; m_{11} are exactly equal to their calculated values. Even their infinitely small variations (that are due to rounding off errors during exclusion of variables x_1 and x_2, for example) at once will lead to a gross (even qualitative) error: instead of three values of parameter λ we see four values. And the four value of parameter λ depends on coefficients variations in system (123).

For a system of equations (123) a problem on the computation of values of parameter λ that help to achieve nonzero solutions when a_{31} or a_{44} is incorrectly posed and the exclusion of variables x_1 and x_2 from system (118) is an example of a transformation that is equivalent in a classical sense and is not equivalent in a widened one. This transformation has changed the correctness of our problem.

It must be concluded that correctness change during equivalent (in a classical sense) transformations can occur not only for differential equations but for simple algebraic systems as well. It is not difficult to check that correctness change will occur in systems of equations of the form (119) but of higher degree as well.

Now let us examine the consequences. If we solve a problem on the search of parameter λ values with the help of which we obtain nonzero solutions for system (118) and to similar systems by means of successful exclusion of variables x_1, x_2 etc. then even infinitely small rounding off error will lead to quite a mistaken result. If the cause is known then for system (118) it is quite easy to overcome this difficulty: First of all it is necessary to exclude variable x_4. By using the last equation from (118) it is necessary to express x_4 by x_1, x_2, x_3 and to introduce it into the remaining three equations. Successive exclusion of x_1 and x_2 will not already change problem correctness. For system (118) everything is simple. But in order to choose the right way for solving equations systems of higher degree it is necessary to know about a possible correctness loss and to be able to overcome troubles that arise legitimately.

In particular it is not always harmless to use a method of transfer from a system of n equations of the second degree (that is widely used) to a first order system of $2n$ equations (to a Hamilton system of equations etc.) Note that a system of n equations of the second degree directly issue from Lagrange equations of a second type. Such a transformation is equivalent in a classical sense, surely there is no doubts in this. But will it be always equivalent in a widened sense (including a case if there are holonomic relations — relations that do not contain derivatives — between variables) — is a statement that needs checking.

But it is quite another matter when a transfer to other number of variables can change a physical sense of a mathematical model. Everything becomes more complex. In §3 we have already come upon a special case of equations system (118) applied to the examination of control systems (equations (33)–(35) and (25)–(26) etc., characteristic polynomials (28), (30) and (31) etc.) and then we saw that a simple exclusion of variable x_4, its expression by

x_1, x_2 and x_3 (on the basis of the last of equations (118)) inevitably lead to mistaken conclusions. Note that here the last of the equations (118) corresponds to equation (35). Really if only variable x_1 can be directly used in a feedback channel then only equations (25)–(26) completely correspond to a physical sense of a problem we are examining. Only in them it is taken into consideration that changes of parameters in control objects and in a regulator take place independently from each other. But system of equations (33)–(35) does not reflect this point although both systems of equations are equivalent to each other in a classical sense and if values of coefficients and parameters are invariable both systems quite identically describe processes that occur in a control system.

Therefore during transformations of mathematical models it is necessary to take into account not only equivalence of transformations with unchangeable coefficients and parameters (it has been done for a long period of time) but it is also necessary to take into account a possible loss of equivalence in a widened sense at parameters variations.

Note that phenomena we are dealing with differ from phenomena that have already been investigated earlier of loosing solution exactness if we apply different solution methods. So for example in [23, 24] it is shown that if we successively undertake a sequence of transformations recommended in methods by Gauss, Householder, Givens, A.N. Krillov small (but finite!) rounding off errors can lead to gross error in solutions. These errors will gradually increase in the course of computations. But if a mistake in coefficients tends to zero an error also tends to zero [23, 24].

But in transformations that we examine the change of correctness occur at once at one step and what is more important - already infinitely small rounding off error can lead to a gross qualitative change of a solution.

Thus we are dealing with a new phenomenon (in [1] this fact was already indicated). Note that in a "pure" problem on matrix principal values that was examined in monographs [23, 24] and others equations (119) are examined but not (115), i.e. the possibility of holonomic relations between variables is not taken into account. But while examining equations (115) no such phenomenon of correctness loss if coefficients variations are infinitely small will occur in a general case (as will be shown later). It is possible that just due to this a phenomenon was discovered so late and up to now it has not been examined thoroughly. But this phenomenon plays an important role in securing reliability of solutions in quite different equations and thus this phenomenon also effects the work of our modern techniques: it must be reliable and work without wrecks. Therefore a phenomenon described and found in works [1, 3, 4] is of great practical value and it must be seriously and thoroughly examined.

Let us consider the problem in a general case, i.e. for a system of arbitrary degree:

$$\begin{cases} (a_{11}-\lambda)x_1+\ldots+a_{1;n-1}x_{n-1}+a_{1;n}x_n=0 \\ \qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1;1}x_1+\ldots+(a_{n-1;n-1}-\lambda)x_{n-1}+a_{n-1;n}x_n=0 \\ a_{n;1}x_1+\ldots+a_{n;n-1}x_{n-1}+a_{n;n}x_n=0 \end{cases} \tag{131}$$

Thus we consider a system of n homogeneous linear equations. In all the first $n-1$ equations parameter λ enters. But in the last equation it does not. The following problem is put: to find parameter λ values at which system (131) has nonzero solutions.

Let us suppose that we can solve this problem by means of successive variables exclusions beginning from x_1 and let us examine a case when we remain with two equations with two variables x_{n-1} and x_n. This situation arises before the last exclusion step. On the basis of known rules for the solution of linear algebraic equations systems and by using Cramer's formula it is not difficult to conclude that these two equations will become:

$$\begin{cases} A_1x_{n-1}+A_2x_n=0 \\ A_3x_{n-1}+A_4x_n=0 \end{cases} \tag{132}$$

and A_1, A_2, A_3 and A_4 will be polynomials of variable λ and they will be equal to the following determinants of $n-1$ order:

$$A_1=det\begin{Vmatrix} (a_{1;1}-\lambda) & \ldots & a_{1;n-1} \\ \ldots & \ldots & \ldots \\ a_{n-1;1} & \ldots & (a_{n-1;n-1}-\lambda) \end{Vmatrix} \tag{133}$$

$$A_1=det\begin{Vmatrix} (a_{1;1}-\lambda) & \ldots & a_{1;n} \\ \ldots & \ldots & \ldots \\ a_{n-1;1} & \ldots & a_{n-1;n} \end{Vmatrix} \tag{134}$$

Therefore a determinant (133) consists of coefficients from the first $n-1$ equations of system (131) in its first $n-1$ variables from x_1 up to x_{n-1}. Determinant (134) will be equal to the same determinant (133) but in which the last column is changed by coefficients column that are situated in system (131) before a variable x_n. For polynomials A_3 we have singular relations:

$$A_1 = det \begin{Vmatrix} a_{2;1} & (a_{2;2} - \lambda) & \dots & a_{2;n-1} \\ \dots & \dots & \dots & \dots \\ a_{n;1} & a_{n;2} & \dots & a_{n;n-1} \end{Vmatrix} \tag{135}$$

i.e. determinant (135) consists of coefficients situated in equations (131) beginning from the first line and up to the last one before variables with indexes from x_1 up to x_{n-1} and polynomial A_4 is determined by means of equality

$$A_1 = det \begin{Vmatrix} a_{2;1} & (a_{2;2} - \lambda) & \dots & a_{2;n} \\ \dots & \dots & \dots & \dots \\ a_{n;1} & a_{n;2} & \dots & a_{n;n} \end{Vmatrix} \tag{136}$$

i.e. determinant (136) differs from determinant (135) in the following: the last column in it is changed by coefficients column in variable x_n.

If we decompose determinants by minors of corresponding lines it is not difficult to write members with the highest degrees of parameter λ. We obtain

$$\begin{aligned} A_1 &= (-1)^{n-1}\lambda^{n-1} + \dots \\ A_2 &= (-1)^{n-2}a_{n-1;n}\lambda^{n-2} + \dots \\ A_3 &= (-1)^{n-2}a_{n;1}\lambda^{n-2} + \dots \\ A_4 &= (-1)^{n-3}(a_{n-1;n}a_{n;1} - a_{n-1;1}a_{n;n})\lambda^{n-3} + \dots \end{aligned} \tag{137}$$

where members with lower degrees of parameter λ are denoted by dots. If we exclude variable x_{n-1} from system (132) we obtain the following equation

$$(A_2A_3 - A_1A_4)x_n = 0 \tag{138}$$

System (132) and thus an initial system (131) can have nonzero solutions for such values of λ at which an expression in parenthesis in formula (138) is equal to zero. If we write only eldest members from (137) we have

$$A_2A_3 - A_1A_4 = a_{n-1;1}a_{n;n}\lambda^{2n-4} + (a_{n-1;n}a_{n;1} - a_{n-1;n}a_{n;1})\lambda^{2n-4} + \dots \tag{139}$$

where by dotes members of degree λ^{2n-5} and of lower degrees are denoted. Now let us examine a case when $a_{n-1;1}a_{n;n} = 0$ (this means that either $a_{n-1;1} = 0$ or $a_{n;n} = 0$) In this case a degree of polynomial (139) turns out to depend on infinitely small coefficients variations of polynomials A_2, A_3, A_4

and thus a number of parameter λ principal values (as well as in a special case that we have examined earlier if $n = 4$) depends on numbers of principal values of parameter λ. This means that a problem on parameter λ principal values computation (i.e. values at which nonzero solutions are possible) turns out to be incorrectly posed.

Note the following delicate aspect: variations of initial coefficients a_{ij} from system (131) do not effect the number of principal values. Really let coefficient $a_{n-1;n}$ change little and become equal to $a_{n-1;n}(1+\varepsilon)$ and coefficient $a_{n;1}$ become equal to $a_{n;1}(1+\delta)$ (if ε and δ are any) then the difference

$$a_{n-1;n}(1+\varepsilon)a_{n;1}(1+\delta) - a_{n-1;n}(1+\varepsilon)a_{n;1}(1+\delta)$$

is all the same remains equal to zero and on a degree of polynom (139) coefficients a_{ij} variations seem not to effect. This delicate aspect became a basis for statements that the problem on defining parameter principal values with the help of successive variables x_i exclusion in system (131) is correctly posed. But in fact if we thoroughly analyse the situation it became clear that the first of coefficients $a_{n-1;n}a_{n;1}$ in equality (139) has come into it from polynomials A_2 and A_3 and the second one — from polynomial A_4. Variations of these coefficients (that are due to rounding off mistakes) can by no means depend on each other. And thus their difference that stands in the parenthesis in formula (139) can be not equal to zero. And as soon as it will not be equal to zero (even if it remains infinitely small) then a degree of polynomial (139) will increase and it will have an additional root. This means that the problem on determination of principal values for system (132) is incorrectly posed.

If the product $a_{n-1;1}a_{n;n} \neq 0$ then in this case (as formula (139) shows) variations of coefficients A_1, A_2, A_3 A_4 do not lead to the change of a degree in polynom (139) and the problem on the definition of parameter principal values is correctly posed.

Now let us examine a classical problem on the definition of principal values of matrix A that can be reduced to the search of such parameter λ principal values at which system (115) has nonzero solutions (as we know). In a classical problem of defining principal values parameter λ enters into all equations and therefore the last equation from equations (131) will be of the form:

$$a_{n;1}x_1 + \ldots + (a_{n;n} - \lambda)x_n = 0 \qquad (140)$$

Polynomials A_1, A_2, A_3 after the last equation from equations (131) is changed by (140) will preserve its form and polynomial A_4 will be equal to

$$\bar{A}_4 = (-1)^{n-1} a_{2;1} \lambda^{n-1} + \dots \tag{141}$$

where members of lower degrees are denoted by dots. In this case a determinant of a system (132) will have the form:

$$A_2 A_3 - A_1 A_4 = a_{2;1} \lambda^{2n-2} + \dots \tag{142}$$

and no loss of correctness in a general case will occur. Now it is clear why the phenomenon of changing correctness in our problem during equivalent transformations has been found out quite recently: in a classical and well-examined problem on the computation of matrix principal values (see [23, 24]) it did not occur.

Note that formally correctness loss can be easily avoided if for example in system (131) variable x_4 can be expressed by means of other variables by using the last equation from equations system and then to introduce this expression into other equations. After reduction of similar members we shall come to a classical problem on the computation of principal values for matrixes of dimension $(n-1) \times (n-1)$. And then we shall not come across with correctness change in the course of solution.

But in a series of cases (and first of all for control systems) such an exclusion of the last variable does not correspond to a physical sense of the problem. And besides it distorts the picture of true changes in a system parameters during the exploitation since parameters of a control object and a regulator (circuits of a feedback) as we know can change independently from each other. In these cases a possible correctness loss (and thus — a possible change of stability preservation property in a closed control system during parameters variations) must be accounted by all means. In preceding §§we have already spoken about it.

Once more let us note: if correctness change possibility of the above problem during equivalent (in a classical sense) transformations of equations is perceived and is taken into account then the possibility of an error due to the correctness change can be easily removed. Only unexpected encounter with the correctness change is dangerous. Only blind and thoughtless belief that if a transformation is equivalent then nothing can change. Solutions themselves during equivalent (in a classical sense) transformations will really not change but the correctness of a problem that we are solving can change. We must not forget this and we must take into account this statement.

Methods based on the construction of degrees matrix.

Let us consider a methods that allows us to easily and quickly find necessary conditions for correctness change during the search of principal values of parameter λ for systems of equations (119) by means of variables successive exclusion.

Let us introduce a conception of "degrees matrix", i.e. a matrix whose elements are degrees of polynomials in a variable λ that stand in corresponding cells of matrix $A - \lambda\bar{E}$ or of any other matrix of polynomials, i.e. matrix whose elements are polynomials.

So for systems (118) degrees matrix is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{143}$$

For system of equations (122) a matrix of degrees is of the form:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{144}$$

for system (123) it has a form

$$\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \tag{145}$$

Now let us examine how will matrix of degrees change in the process of variables exclusion. If we exclude variable x_1 from the first and second equation from system (131) then corresponding lines of degrees matrix for this system are of the form:

$$\begin{matrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \end{matrix} \tag{146}$$

In order to exclude x_1 we must multiply the second line by $a_{11} - \lambda$ and the first one by a_{21}. After the multiplication the lines of degree matrix will take the form:

$$\begin{matrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 1 \end{matrix} \tag{147}$$

(i.e. the first line of a degrees matrix remained unchanged and all elements in the second line increased by 1). Later in order to exclude x_1 it is necessary to exclude the second line from the first one and here a new line appears in which a coefficient in x_1 is equal to zero. Thus in degrees matrix one new line that is shorter than the previous one by 1 appears instead of two lines (147). The first element disappears and all numbers of this new line will correspond to the largest number of the first and the second line. Thus initial lines (146) will turn into a line

$$\begin{matrix} 2 & 1 & 1 & \dots & 1 \end{matrix} \tag{148}$$

If we apply this simple rule for the formation of a new line from each pair of lines in an initial matrix it will not be difficult to see that a line $2 \quad 1 \quad 1$ will appear instead of the first and second line of initial degrees matrix a line $0 \quad 1 \quad 0$ will appear instead of the third and the fourth lines. And so after the exclusion of a variable x_1 the following transfer will occur:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

i.e. — the transfer from matrix (143) to matrix (144). Earlier we have made this conclusion by means of direct computation during exclusion of x_1. After exclusion of variable x_2 the following transfer will occur by applying the same rules:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

We can easily make a conclusion whether the loss of correctness can take place after the exclusion of two variables. Here a remaining matrix of dimension 2×2 will help us. The loss of correctness can occur if the sums on diagonals are equal in a matrix of degrees (of dimension 2×2). In matrix (145) they are just equal. This means that the reduction of higher degrees and the loss of correctness are possible. Direct computation that has been carried out earlier for equations system (118) confirms this.

We must note: the examination of degrees matrix only speaks about the possibility (or impossibility) of correctness change. And it gives only necessary but not sufficient conditions for the change of correctness as an earlier examined example of system (118) confirms. The loss of correctness takes place only if $a_{31}a_{44} = 0$. But just necessary conditions are most important in practice.

As an example let us consider the following aspect: is the loss of correctness possible in a classical problem on the definition of principal values for a matrix (dimension 4×4) that corresponds to the problem of the search of parameter λ principal values at which nonzero solutions of the following equations system are possible:

$$\begin{cases} (a_{11} - \lambda)x_1 + \ldots + a_{14}x_4 = 0 \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{41}x_1 + \ldots + (a_{44} - \lambda)x_4 = 0 \end{cases} \tag{149}$$

for which degrees matrix is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{150}$$

If we transform this matrix according to the above simple rule then we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix} \tag{151}$$

In a last matrix (dimension 2×2) sums on a diagonal are not equal. It means that in a classical problem on the principal values search for matrix 4×4 correctness loss in a general case is impossible during the exclusion of two variables. This corresponds to the earlier statement: in a classical problem on the computation of principal values of matrixes having any dimensions no correctness loss in a general case occurs.

Now let us examine a case of five homogeneous equations system in the first three of which parameter λ enters but it does not enter into the last two equations. This system is of the form:

$$\begin{cases} (a_{11}-\lambda)x_1 + \ldots + a_{15}x_5 = 0 \\ \quad \ldots\ldots\ldots\ldots\ldots\ldots \\ \quad a_{51}x_1 + \ldots + a_{55}x_5 = 0 \end{cases} \tag{152}$$

Degrees matrix for a system (152) is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{153}$$

If we transform it according to the above rules we obtain:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}$$

In the last matrix sums on diagonals are equal. This means that when we exclude three variables from a system (152) then a problem on the computation of principal values of parameter λ from a remaining system of two equations with two variables can turn out to be incorrectly posed.

The methods of computation of principal numbers of degrees matrix allows us to easily solve quite different problems on the checking of correctness change for very different systems of linear homogeneous equations with parameter λ.

Method of calculation of degrees matrixes allows us to easily answer the question of a possible solution correctness even for such a case when (after excluding several variables) three equations with three variables remained.

Let us come to the following degrees matrix (dimension 3×3) after exclusion of several variables:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{155}$$

whose determinant is equal to

$$\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - \\ - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} \quad (156)$$

as is well-known.

For degrees matrix each of six triplex products will be equal to some member (to an index of variable λ corresponding polynomial). The problem on the computation of principal values of λ is correctly posed in such case if there is only one the largest number among these six members. In this case during parameters variations a degree of the polynomial will not apriori change in a general case and thus a number of principal values will not also change. But if there are two or more the largest same numbers among six numbers entering into determinant (156) then during parameters variations a degree of the polynomial can change and a problem on the computation of principal values can become incorrectly posed.

Here is an example. Let us consider a classical problem on the computation of principal values for a matrix (dimension 4×4). It corresponds to the problem on the search of such values of λ for which nonzero solutions of the above equations system (149) exist. After the exclusion of one variable we shall have a degrees matrix:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (157)$$

for which a determinant (156) will be equal to

$$\Delta = 5; \quad 2; \quad 3; \quad 3; \quad 4; \quad 3; \quad (158)$$

(we write only indexes). Among six numbers that we have obtained the largest is only one this means that in a classical problem on the computation of principal values after the exclusion of one variable no loss of correctness takes place.

But if we examine a system of equations (152) when parameter λ enters into three first equations and does not enter into two last ones then after exclusion of two variables we shall come (as a relation (154) shows) to the following degrees matrix

$$\begin{pmatrix} 3 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad (159)$$

for which a determinant (156) will be equal to

$$\Delta = 5; \quad 4; \quad 4; \quad 4; \quad 4; \quad 5;$$

this means that after exclusion of two variables we may come to an incorrectly posed problem.

The examination of degrees matrixes shows that the simplest cases of correctness change in the problem that we are solving are possible even in systems consisting of three equations.

Let us consider the following quite simple system as an example:

$$\begin{cases} \lambda x_1 - x_3 = 0 \\ x_1 - 2\lambda x_2 + x_3 = 0 \\ x_1 - x_2 = 0 \end{cases} \tag{160}$$

for which let us consider the same problem on the search of values in parameter λ at which nonzero solutions are possible. Let us multiply the second equation by λ (it is lawful since $\lambda = 0$ is not the solution — as can be easily checked) and let us add it to the first equation. A variable x_1 will be excluded and we shall obtain:

$$2\lambda^2 x_2 - (\lambda + 1)x_3 = 0. \tag{161}$$

If we subtract the third equation from the second one we shall have

$$(1 - 2\lambda)x_2 + x_3 = 0. \tag{162}$$

If we exclude x_2 from equations (161) and (162) we shall obtain

$$(1 - \lambda)x_3 = 0, \tag{163}$$

hence we find the only principal value $\lambda_1 = 1$.

But a problem of exclusion x_2 from equations system (161) and (162) is not correctly posed. If in equation (162) a coefficient that stands before variable x_3 is not equal to 1 but to $1 + \varepsilon$ then after exclusion of x_2 we obtain the following equation instead of equation (163)

$$(2\varepsilon\lambda^2 - \lambda + 1)x_3 = 0 \tag{164}$$

from which we shall find two principal values λ_1 and λ_2 and λ_2 does not tend to 1 if $\varepsilon \to 0$.

At the same time if we begin from the third equation from (160) and introduce a value $x_1 = x_2$ obtained from it into the first two equations then we shall come to a system

$$\begin{cases} \lambda x_2 - x_3 = 0 \\ (1 - 2\lambda)x_2 + x_3 = 0 \end{cases} \tag{165}$$

for which a problem on the computation of a principal value of λ is correctly posed.

Therefore the correctness or incorrectness of the problem can depend on the solution method.

Here is another example: a system

$$\begin{cases} (1 - \lambda)x_1 + x_2 + 2x_3 = 0 \\ x_1 + (1 - \lambda)x_2 + 3x_3 = 0 \\ x_1 + x_2 = 0 \end{cases} \tag{166}$$

For it the only value of a parameter λ at which nonzero solutions are possible is $\lambda = 0$. And its determination problem is correctly posed if we start finding a solution from the last equation. If we exclude variables according to the order of their indexes (that is natural if we make machine computations) then after exclusion of x_1 with the help of multiplying the second equation from (166) by $(1 - \lambda)$ (note that $\lambda = 1$ is not the solution) and subtracting of an obtained line from the first equation then we shall obtain

$$(\lambda^2 - 2\lambda)x_2 + (1 - 3\lambda)x_3 = 0. \tag{167}$$

If we subtract the third equation from the second one of (166) we shall have

$$\lambda x_2 - 3x_3 = 0. \tag{168}$$

By exclusion x_3 from equations (167) and (168) we shall obtain

$$5\lambda x_2 = 0, \tag{169}$$

hence we at once conclude that the only principal value is $\lambda = 0$. Therefore in relation to the definition problem of principal values of parameter λ a system of equations (167) and (168) is equivalent (in a classical sense) to a system (166). But for system (167)–(168) the computation problem of parameter λ principal values is incorrect: for example in equation (168) a coefficient in

x_3 is equal not to 3 but to $3(1+\varepsilon)$ then after exclusion of x_3 from equation (167) and (168) we shall obtain the following equation instead of equation (169)

$$(3\varepsilon\lambda^2 - 6\varepsilon\lambda - 5\lambda)x_2 = 0 \qquad (170)$$

from which we shall find two principal values: $\lambda_1 = 0$, $\lambda_2 = 2 + \frac{5}{3\varepsilon}$.

This example clearly shows that infinitely small parameters variation, infinitely small rounding off error at computation can lead to a gross error and that the second (false) principal value of λ if $\varepsilon \to 0$ by no means tends to λ_1 and it disappears only when $\varepsilon = 0$. Even if ε is very small the second value λ_2 is large.

Examples with systems of three equations (160) and (166) are the simplest. We can start the story about the correctness change from them. We can as well begin the story about all interesting consequences of its change to the children during the work at mathematical circles.

For control systems (in equations of which one of variable is control) the loss of correctness means the loss of stability during parameters variations and can become the source of dangerous wrecks and catastrophes. We have spoken about the loss of correctness for control systems is essential for fourth degree systems as it was shown in previous §§. Specific problems that occur in control systems is stabilization systems etc. were reflect in supplementary chapter included in the second complemented edition of a monograph [29] edited by St.-Petersburg Technical University (the former Polytechnical Institute) in 1977.

In [29] an interesting question on the relation between the description of control systems in the language of structures schemes and in the language of differential equations was not considered. Usually these languages of description are considered to be equally complete and in practice (as a rule) firstly a structural scheme of a projected system is constructed that reflects the interaction of its separate elements. And later on its basis a system of differential equation is written whose solution is obtained with the help of computing technique and we follow the obtained solutions. But in fact the language of structures schemes reflects phenomena and processes more completely that occur in a real system than a language of differential equations especially if we take into account an inevitable small drift of parameters in practice.

Now let us return to a control object (33) that we have earlier examined.

Its controlling interaction is formed according to equation (35). We mean a variable x_2. In the language of structures schemes system (33)–(35) will look in such a way as it is shown on figure 1. A structures scheme shown on figure 1 says that a controlling interaction x_2 in a feedback channel is formed from variables x_1, x_3 and x_4 with intensity coefficients -1, -2, -1. If variables x_3 and x_4 are inadmissible for direct measure and usage in a feedback channel and we are eager to secure the same transient processes that occurred in systems (33)–(35) then we can express variables x_3 and x_4 by means of admissible variables x_1 and x_2 and their derivatives by applying equations (33). Having made such a change of variables we shall obtain instead of a feedback (a regulator) described by equation (35) a regulator (a feedback) that is equivalent to it. But it is described by equation (26)In a language of structures schemes a control object (33) with a feedback (26) will look as it is shown on figure 2. Note that if parameters values in a control object and a regulator are unchanged and correspond to calculated ones then structures schemes shown on figs. 1 and 2 are equivalent and transient processes described by formula (29) equally correspond to them. But if in a structure scheme shown on fig. 1 we change coefficients of regulator (35) by small values — i.e. if we change coefficients (-1), (-2) and (-1) by small values in a feedback channel then a transient process changes little and stability of a closed system will be preserved. If we change even infinitely little some of coefficients in a channel of a feedback for a structure scheme shown on fig. 2 then transient processes can essentially change and a closed system can become unstable. In the language of structures schemes all these phenomena can be described simpler and more evident than in the language of differential equations since in a structure scheme it is especially clear that parameters variations in the circuit of a feedback can be independent of a control object parameters variations. Besides it is evident that although structure schemes shown on fig. 1 and fig. 2 describe the same transient processes in a real system (if parameters and coefficient invariable) but at the same time they are not identical to each other and therefore at parameters variations they behave themselves differently. Thus we can better understand why equations (25)–(26) and (33)–(35) are completely equivalent to each other in a classical sense and at the same time they are not equivalent in a widened sense.

Now let us examine a question of differential equations solutions behavior on a phase plane. It is well-known that it is convenient to examine a qualitative picture of differential equations solutions behavior on a phase plane — i.e. a plane where one of variables is put on abscissas axis and another variable is put on an ordinates axis (or a derivative of the first variable).

Let us consider the following system for an example:

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\omega_0^2 x_1. \end{cases} \tag{171}$$

This system can be easily integrated and we can find the solutions:

$$\begin{cases} x_1 = c_1 \sin(\omega_0 t + c_2) \\ x_2 = c_1 \omega_0 \cos(\omega_0 t + c_2), \end{cases} \tag{172}$$

where c_1 and c_2 — integration constants that depend on initial conditions.

If we exclude variable t (time) we shall find equations of movement trajectories on a phase plane:

$$\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_1^2 \omega_0^2} = 1. \tag{173}$$

It is directly evident that equation (173) is an equation of a family of similar ellipses that are put into each other only one ellipse corresponding to a certain initial condition passes through each point of a phase plane. From equation (173) periodicity of solutions, their limitedness etc. follows. But at the same time (and it is very important) these properties of solutions can be obtained and considered without a direct integration of initial equations (171). Really let us divide the second equation from (171) by the first one. We shall have the following equation

$$\frac{dx_1}{dx_2} = -\omega_0^2 \frac{x_1}{x_2}, \tag{174}$$

it is a first degree differential equation. It is easier to integrate this equation than system (171). By integrating equation (174) with the help of variables separation we shall obtain an equation (173) with the help of which we can have a qualitative picture of solutions (limitedness, periodicity etc.) and without integrating equations (171) themselves.

The same method can be used (and is widely used) for the examination of solution behavior in nonlinear equations that can not be integrated in elementary functions.

Therefore a phase plane method for the formation of a phase portrait is widely used during the investigation of quite different objects whose mathematical model are systems of linear and nonlinear differential equations. There is an immense number of monographes and articles devoted to different aspects of a phase plane methods and its applications.

In order to correctly apply a phase plane methods it is necessary to search conditions at which a qualitative picture of trajectories behavior on a phase plane would not substantially change during small coefficients changes of their right sides. These small changes are quite inevitable in real conditions.

In a classical monograph [30] the following correct statement was already given: "there is no factor that we have taken into account that can remain absolutely unchanged" and "parameters can not be considered absolutely constant in real physical system but only approximately constant" and "therefore we cannot at once refuse to examine such qualitative aspects of movement that disappear during small changes in the type of differential equations describing a system".

Therefore let us thoroughly consider a possibility of a qualitative change in a phase portrait, in a qualitative portrait of trajectories behavior on a phase plane during the solution of equations, in the course of transformations undertaken during the solution of equations.

Let us examine a system of equations:

$$\dot{x}_1 = x_1 + x_2 + 2x_3 = 0 \tag{175}$$

$$\dot{x}_2 = x_1 + x_2 + 3x_3 = 0 \tag{176}$$

$$x_1 + x_2 = 0 \tag{177}$$

This system is solved in a quite elementary way if we introduce (176) into (175) and then after it if we subtract one from the other. We shall obtain

$$x_3 = 0; \qquad x_2 = c_1 \qquad x_1 = -c_1. \tag{178}$$

Now let us see what will we have if we apply a conventional solution method by means of successive exclusion of variables, beginning from x_1. If we multiply equation (176) by an operator polynomial $D - 1$ (and if we check that function $x_1 = c_0 e^t$ that correspond to $D - 1 = 0$ is not a solution) and subtract it from equation (175) then we shall obtain the following equation that does not contain x_1:

$$(D^2 - 2D)x_2 + (1 - 3D)x_3 = 0. \tag{179}$$

If we subtract a sum $x_1 + x_2$ that is equal to zero from equation (176) we shall obtain the second equation with variables x_2 and x_3:

$$Dx_2 = 3x_3. \tag{180}$$

By excluding a variable x_3 from a system of equations (179)–(180) we shall obtain

$$5Dx_2 = 0 \tag{181}$$

hence we shall find that $x_2 = c_1$; $x_3 = 0$; $x_1 = -c_1$ — i.e. that we shall find the same solutions as before. This once more confirms the opinion that a system of equations (179), (180), (177) is equivalent to system (175), (176), (177) since we have used only equivalent transformations. But system (179), (180), (177) is equivalent to system (175), (176), (177) only in a classical sense but not in a widened one. During variations of coefficients systems behave themselves differently. The problem on the search of solutions $x_1(t)$, $x_2(t)$, $x_3(t)$ for a system of equations (179), (180), (177) is incorrectly posed.

Really let a coefficient in x_3 from equation (180) be not a 3 but $3(1+\varepsilon)$ where ε — number that is small in comparison to 1. Then after exclusion of x_3 we shall come to equation

$$[3\varepsilon D^2 - (5 + 6\varepsilon)D]x_2 = 0 \tag{182}$$

hence if we solve it we shall obtain

$$x_2 = c_1 + c_2 e^{(2+\frac{5}{3\varepsilon})t} \tag{183}$$

$$x_3 = \left(\frac{6\varepsilon + 5}{9\varepsilon + 9\varepsilon^2}\right) c_2 e^{(2+\frac{5}{3\varepsilon})t} \tag{184}$$

We know now that during infinitely small variations of coefficients in equations (179) or (180) or during infinitely small rounding errors while we successfully undertake variables exclusion from initial system (175)–(176) solutions of equations and the whole character of a phase portrait, of trajectories behavior on a phase plane change substantially.

Let us consider trajectories of equations (175)–(177) solutions on a phase plane where on ordinates axis values of variable x_2 and on an abscissas axis values of variable x_3 are set. If we take into account formulas (178) we may conclude that a phase portrait of equations (175)–(177) solutions is of the form, shown on figure 3 - i.e. it consists of a set of dots $x_2 = c_1$ that fill all ordinates axis (on fig. 3 separate dots are symbolically shown). Solutions of the equations (179), (180), (177) that are equivalent (in a classical but not in a widened sense) to equations (175)–(177) will have the same phase portrait. But if in equation (180) coefficient in x_3 is not equal to 3 but to $3(1+\varepsilon)$ and $\varepsilon \neq 0$ then the character of phase trajectories and the whole phase portrait substantially changes: formula (183)–(184) shows that now

$$x_2 = c_1 + \frac{9\varepsilon + 9\varepsilon^2}{6\varepsilon + 5} x_3 \tag{185}$$

First of all let us examine a case $(2 + \frac{5}{3\varepsilon}) > 0$. In this case exponents indexes in formulas (183)–(184) are positive and with the increase of variable t (time) solutions $x_1(t)$ and $x_2(t)$ will unlimitedly increase. Phase trajectories will be straight lines that go up and to the right. The incline of these lines depends on ε. On figure 4 a phase portrait for one of values of ε is shown.

Now let us turn to a case when $(2 + \frac{5}{3\varepsilon}) < 0$ and therefore indexes of exponents in formulas (183) and (184) are negative. In this case with time increase $x_3 \to 0$, $x_2 \to c_1$ if we take into account (185) phase trajectories will take another form: although they remain straight lines that are filling the whole phase plane now they tend to ordinates axis from the left and from the right as well and end on it (as is shown on figure 5).

We see that during transformations of equations that are equivalent in a classical sense but not in a widened sense solutions phase portraits (if we take into account inevitable variations) and a character of phase trajectories itself can essentially change.

Therefore during the examination of different objects and systems on a phase plane it is necessary to watch with special care: that transformations of systems are used. It is necessary to distinguish between transformations that are equivalent either in a classical or in a widened sense from transformations that are equivalent in a classical sense but not in a widened sense. While examining stability of a character of a phase portrait in relation to small changes of coefficients, parameters etc. we can advise you to undertake examinations for all used forms of differential equations. As a matter of fact already in §3 during the analysis of equations system (25)–(26) we have seen that at variations of some coefficient of this system a character of a phase portrait will sharply change. At calculated values of parameters all phase trajectories tend to the origin of coordinates in the course of time. And as during infinitely small variations of some coefficients variables $x_1(t)$ and $x_2(t)$

can unlimitedly increase when $t \to \infty$. But we shall not see this important property of a phase portrait if we rewrite equations (25)–(26) to a normal Cauchy form (33)–(35) (as it is very often done) although equations (25)–(26) and (33)–(35) are equivalent (in a classical sense) and at calculated values of coefficients these equations have the same solutions: $x_1(t)$ and $x_2(t)$.

Therefore if we do not take into account that there is a difference between transformations that are equivalent in a widened sense and transformations that are equivalent in a classical sense but not in a widened sense we will come to erroneous results in different fields. In particular — during examinations of phase portrait as well. Naturally errors in the estimate of stability resources can be called the most dangerous errors. They were considered in §§3–11. It is so because these errors can become causes for dangerous wrecks.

Besides (as it was shown above) the usage of such transformation can become an additional cause of mistakes in different computations. In this book we have given the following example of this — in a generalized problem on the computation of principal values. But if is quite probable that such mistakes occur during other calculations and computations. There is a wide field for further scientific investigations here.

Comparison of different methods of variables exclusion.

Earlier we have examined the simplest method of excluding variables from a system of linear homogeneous equations (119). It was a method of successive exclusion with the help of multiplications and additions.

If we investigate systems consisting of four equations with four variables by this method we shall find that after the exclusion of two variables (if $a_{31} = 0$ or $a_{44} = 0$) the loss of correctness in our problem on the finding of parameter λ principal values occurs. But if $a_{31}a_{44} \neq 0$ correctness loss does not occur.

Previously we have principally examined such systems in which $a_{31}a_{44} = 0$. Now let us examine a case if $a_{31}a_{44} \neq 0$.

Let us return to control systems and let us examine a system consisting of a control object:

$$\begin{cases} \dot{x}_1 = x_2 + x_3 + u \\ \dot{x}_2 = x_1 + x_3 + u \\ \dot{x}_3 = x_1 + x_2 + 2u \end{cases} \tag{186}$$

and a regulator

$$u = -x_1 - x_2 - x_3. \tag{187}$$

If we introduce (187) into (186) we shall obtain equations of a closed system:

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -x_2 \\ \dot{x}_3 = -x_1 - x_2 - 2x_3. \end{cases} \tag{188}$$

A characteristic polynomial of a closed system is equal to a determinant

$$\begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 1 & 1 & \lambda+2 \end{vmatrix} = \lambda^3 + 4\lambda^2 + 5\lambda + 2 = (\lambda+1)^2(\lambda+2) \tag{189}$$

and it has roots $\lambda_1 = -2$, $\lambda_2 = \lambda_3 = -1$ that lie in the left half-plane that is far from an imaginary axis.

The closed system is stable.

Now let us suppose that only variable x_3 can be directly measured and used in a feedback channel. Let us transform equations of a control object and a regulator to variables x_3 and u by excluding variables x_1 and x_2 during additions and multiplications. If we use the above obtained formulas (133)–(136) it is not difficult to compute:

$$A_1 = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \lambda^3 + 3\lambda + 2,$$

$$A_2 = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2\lambda^2 + 2\lambda,$$

$$A_3 = \begin{vmatrix} 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \\ 1 & 1 & 1 \end{vmatrix} = \lambda^2 + 2\lambda + 1,$$

$$A_4 = \begin{vmatrix} 1 & -\lambda & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = -(\lambda+1).$$

Equations of control object (136) and a regulator (137) will take the form

$$\begin{cases} (D^3 - 3D - 2)x_3 = 2(D^2 + D)u \\ (D^2 + 2D + 1)x_3 = (D + 1)u \end{cases} \tag{190}$$

in variables x_3 and u.

A characteristic polynomial of a closed system is equal to the following determinant:

$$\begin{vmatrix} -\lambda^3 + 3\lambda + 2 & 2\lambda^2 + 2\lambda \\ \lambda^2 + 2\lambda + 1 & -\lambda - 1 \end{vmatrix} = \lambda^4 + 5\lambda^3 + 6\lambda^2 + 7\lambda + 2 = (\lambda+1)^3(\lambda+2). \tag{191}$$

We see that if we compare it with a characteristic polynomial (189) one more root have appeared — $\lambda_4 = -1$ that is not a result of parameters variations and that does not depend of them. This means that system (190) is not completely equivalent (in a classical sense) to a system (186)–(187).

In the process of multiplication and addition we have introduced a redundant root. In this special case this root can be easily removed: the second equation from (190) can be reduced by an operator polynomial $(D + 1)$ and we shall come to the following equations for a control object and a regulator:

$$(D^3 - 3D - 2)x_3 = 2(D^2 + D)u \tag{192}$$

$$(D + 1)x_3 = u \tag{193}$$

System (192)–(193) has the same order as an initial system (186)–(187).

It is not difficult to make the following conclusion: system (190) and system (192)–(193) both preserve stability at variations of any its coefficients (the same can be said about system (186)–(187)). System (192)–(193) has the same characteristic polynomial as system (186)–(187) and both systems are equivalent between themselves as in a classical sense so in a widened one. There has not occurred correctness loss during the use of transformations.

But the presence of the same operators multiplier whose reduction is possible is a rare special case. This operators multiplier is in the left and the right sides of a regulator equation. But in a general case when variables x_1 and x_2 are excluded according to the simplest methods that we have earlier described from equations of a control object and a regulator of the type (186)–(187) but with another coefficient we come (if $a_{31}a_{44} \neq 0$) to a system whose characteristic polynomial has a redundant fourth root in comparison with an initial system (and this root naturally does not depend on variations of parameters and on variations of system coefficients). Therefore a transformed system is not wholly equivalent to an initial one in a classical sense and transient processes in an initial and transformed systems will not be identical. Therefore other variables exclusion methods were proposed that could not violate equivalence in its classical sense.

Already in the work [15] such a method for a control object of the third order was examined:

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1 u \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2 u \\ \dot{x}_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_3 u \end{cases} \tag{194}$$

with the following regulator:

$$u = k_1 x_1 + k_2 x_2 + k_3 x_3. \tag{195}$$

A characteristic polynomial of a closed system (194)–(195) will be a third order polynomial.

If only variable x_3 can be directly measured then variables x_1 and x_2 are excluded from equations (194) by means of the above method and as result we shall obtain a well-known equation

$$A_1(D)x_3 = A_2(D)u, \tag{196}$$

where

$$A_1(D) = \begin{vmatrix} a_{11} - D & a_{12} & a_{13} \\ a_{21} & a_{22} - D & a_{23} \\ a_{31} & a_{32} & a_{33} - D \end{vmatrix} \tag{197}$$

and correspondingly:

$$A_2(D) = \begin{vmatrix} a_{11} - D & a_{12} & -b_1 \\ a_{21} & a_{22} - D & -b_2 \\ a_{31} & a_{32} & -b_3 \end{vmatrix} \tag{198}$$

In order to exclude variables x_1 and x_2 from equation (195) it was proposed to multiply the right and the left sides of the first equation from (194) by $r_1 + r_2 D$, the second equation — by $s_1 + s_2 D$, the third one — by $t_1 + t_2 D$ (where all r, s and t — are coefficients that must be definite). After this all three equations are added and equation (195) is added to the sum. As a result we shall obtain a relation between a control u and variables x_1, x_2, x_3 and their derivatives. In this relation coefficients $r_1; \dots ; t_2$ that are as yet unknown were chosen in such a way that coefficients before x_1 and $\dot{x}_1$, x_2 and $\dot{x}_2$ would turn into zero. The conditions of turning into zero of these coefficients led to a system of equations that are necessary for the definition of $r_1; \dots ; t_2$. Finally we shall obtain the following equation of a regulator:

$$W_1(D)x_3 = W_2(D)u, \tag{199}$$

where $W_1(D)$ is a second order polynomial and $W_2(D)$ — a first order polynomial. A characteristic polynomial of a closed system will be equal to

$$W_1A_2 - W_2A_1 \tag{200}$$

and during this exclusion method of variables it will be always a third order polynomial just as in an initial system (194)–(195) (if $a_{31}a_{44} = 0$ both exclusion methods coincide) there will be no redundant roots, members of the fourth order in polynomial (200) will be mutually reduced and transient processes will be the same as in an initial system. But the reduction of members with a fourth order variable D will naturally occur only if all coefficients and parameters of a control object (196) and a regulator (199) will be exactly equal to their computed values. If their parameters variety infinitely little no reduction can occur. And thus (as it was shown earlier) the stability of a closed system can be lost. Equations (196)–(199) are equivalent to equations (194)–(195) in a classical sense but not — in a widened sense.

This exclusion method of variables is cumbersome and M.A. Galaktionov proposed such an exclusion method that is suitable for any number of variables and for any rectangular matrix H that connect vector y of really measured and really used in a circuit of feedback variables with a total vector of variables in a control object. M.A. Galaktionov's method is published in [1], in §§3–4 of the fifth chapter. Therefore we shall speak about his method only in short. The method can be applied to linear control objects of an arbitrary order:

$$\dot{x} = Ax + Bu, \tag{201}$$

where A — constant coefficients matrix (dimension $n \times n$), B — vector-column of coefficients if an interaction is controlling. A control object (201) is closed by a regulator

$$u = kx, \tag{202}$$

where k — matrix-line.

Let only variables y that are connected with variables x by the following matrix

$$y = Hx \tag{203}$$

be directly measured and be directly used in a feedback channel. In a special case matrix H can also be a matrix-line. So in an example with a system (194)–(195) $H = (0, 0, 1)$. Then let us consider a case when $n = 3$ and $H = (0, 0, 1)$. Since a question on the stability of system (201)–(202) and on the stability preservation at parameters variations reduces to the question about parameters principal values λ that is put instead of differentiation operator then let us carry out this change and let us write equation (201) as follows:

$$\lambda x = Ax + Bu. \tag{204}$$

Now let us multiply the left and the right sides of equation (203) by λ and let us introduce instead a product λx its value from (204) then we shall have:

$$\lambda y = HAx + HBu. \tag{205}$$

Let us multiply the left and the right sides of this equality once more by λ and let us again introduce instead of product λx its value from (204) we shall obtain:

$$\lambda^2 y = HA^2 x + HB\lambda u + HABu. \tag{206}$$

Equations (203), (205) and (206) can be considered as a one vector-matrix equation connecting y, x and u:

$$\begin{pmatrix} y \\ \lambda y - HBu \\ \lambda^2 y - HB\lambda u - HABu \end{pmatrix} = \begin{pmatrix} H \\ HA \\ HA^2 \end{pmatrix} x \tag{207}$$

From equation (207) it following that

$$x = \begin{pmatrix} H \\ HA \\ HA^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} y \\ \lambda y - HBu \\ \lambda^2 y - HB\lambda u - HABu \end{pmatrix} \tag{208}$$

where by a symbol

$$\begin{pmatrix} H \\ HA \\ HA^2 \end{pmatrix}^{-1} \tag{209}$$

a matrix that is inverse to the matrix

$$\begin{pmatrix} H \\ HA \\ HA^2 \end{pmatrix} \tag{210}$$

is denoted.

From equation (208) it follows that

$$x = L_1 y + L_2(\lambda y - HBu) + L_3(\lambda^2 y - HB\lambda u - HABu), \tag{211}$$

where L_1, L_2, L_3 are vector-columns and therefore

$$u = kx = KL_1 y + KL_2(\lambda y - HBu) + KL_3(\lambda^2 y - HB\lambda u - HABu). \tag{212}$$

Since in the above case matrix $H = (0, 0, 1)$ and a product of a matrix-line by a vector-column is a number then from formula (212) it follows that (as $y = x_3$) after exclusion of variables x_1 and x_2 regulator (202) becomes:

$$(n_1 D^2 + n_2 D + n_3)x = (n_4 D + n_5)u \tag{213}$$

where $n_1; \dots; n_5$ — numbers which can be computed on the basis of formula (212). The analysis of these numbers shows that in a general case if $b_3 \neq 0$ a closed system can loose stability at infinitely small variations of parameters of a control object or a regulator (213). After the exclusion of variables x_1 and x_2 the loss of correctness of our problem on the stability of a closed system occurred.

Surely a described method of calculating a regulator that used only a real vector of variables on an exit can be applied to control objects of higher orders and to any matrixes (203). Examples are given in [1].

We see that correctness loss during equations transformations is a delicate and complex phenomenon. It can depend on a type of an applied transformations and even on their order. At the same time an unexpected occurrence with correctness change can lead to errors during the analysis of stability preservation in quite different systems and objects and can become the cause of wrecks and catastrophes. The correctness loss can also become one more additional cause of error in computations if there are very small rounding off mistakes during computations.

The phenomenon of correctness loss requires a thorough investigation and analysis in future.

§14. On a third class of problems of mathematics, physics and technique — on problems that are intermediate between correctly and incorrectly posed.

The examples that we have investigated show that in mathematics, physics and technique besides correctly and incorrectly posed problems there is one more class — the third one. It is such a class of problems that changed its correctness during equivalent transformations including transformations used during their solution.

Correctly posed problems have been solved in mathematics since a long period of time. The works of famous French mathematician J. Hadamar (1865–1963) originated the examination of incorrectly posed problems. His first work on this theme was published in 1902 in [22]. In the second half of the 20th century the importance of incorrectly posed problems was found and methods of their regularization and solution were proposed. Russian academician A.N. Tikhonov and his school [21, 22] contributed a large share into world science and they were acclaimed for it.

In [1–5, 23, 29] contours of one more — third class of problems in mathematics, physics and technique were given. This class combines problems that are able to change correctness during equivalent transformations.

This new third class of problem is only on the verge of investigations. The difficulty of solving this class of problems lies in the following: an unexpected correctness change can become a source of errors in computations.

It is extremely dangerous when an initial, primary mathematical model that directly follows from the laws of mechanics and physics — a model of an examined object or a phenomenon (in relation to our problem) is not correctly posed.

Since initial models are often not convenient for examination they are usually transformed and reduced to a "conventional" form for whose investigation we can use a well-developed theory and programs compilation. For the reduction of a mathematical model to a convenient form only equivalent transformations are surely used. Therefore solutions of an initial and a transformed model coincide.

As to correctness usually it checked only once according to a convenient transformed system, silently supposing that as used transformations were equivalent (in a classical sense) the solutions must not change and thus correctness of a solved problem must not change as well. In fact it is not so and during equivalent transformations correctness can change. If our checked transform mathematical model is correct in relation to the problem we are examining then this does not at all speak about the correctness of an initial model. And what is more important this does not speak about a true correctness of a problem that we are considering. And an error in the correctness check can become a cause of wrecks and catastrophes. We have spoken about it already.

Now let us examine some examples.

As the first example we can take a problem (known in the 70-s) on the synthesis of an optimal auto-steering that secures the minimal velocity loss of a ship during its movement in conditions of irregular rough sea. As we know the following equation can serve a mathematical model for the movement of a ship according to its course:

$$(T_1^2 D^2 + T_2 D)\theta = u + \varphi(t) \tag{214}$$

in which θ — an angle of a ship deviation from its courses in degrees, T_1 and T_2 — time constants in seconds, $D = \frac{d}{dt}$ — differentiation operator, u — angle of steering deviation from a diameter, in degrees (it plays the role of control), $\varphi(t)$ — perturbing interaction, a moment of forces from the wind and a rough sea that leads astray the ship from its course. It is measured in degrees of the steering deviation that creates a moment of this same value [16].

Equation (214) — a mathematical model of the ship movement according to its course. It directly issues from equations of theoretical mechanics — equations of movements equilibration in relation to a vertical axis that comes through the center of the ship masses. The right side of equality (214) is equal to a sum of moments in relation to moments in relation to this axis, the left side reflects an inertial moment of the corpse and a damping water effect.

It is well-known that moment $\varphi(t)$ is a stationary accidental process and its spectral density of power can be approximated by an analytical expression:

$$S_\varphi = D_\varphi \frac{1}{(\omega^2 - \alpha^2 - \beta^2)^2 + 4\alpha^2\omega^2} \tag{215}$$

(Rakhmanin-Firssov's spectrum) where D_φ — dispersion, α and β — coefficients that depend on a character of waving. Moment $\varphi(t)$ can be also represented as a process on the output of a linear second order link:

$$(D^2 + 2\alpha D + \alpha^2 + \beta^2)\varphi = \eta(t) \tag{216}$$

in input of which are present a function $\eta(t)$ — perturbing interaction of "white noise" type.

Equation (216) can be transformed into a system of two equations of first degree if we put a new variables $x_1 = \varphi$ and $x_2 = \dot{\varphi}$. Then we have:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(\alpha^2 + \beta^2)x_1 - 2\alpha x_2 + \eta(t) \end{cases} \tag{217}$$

Velocity loss arises due to additional resistance of ship corpse to movement if $\theta \neq 0$. This additional resistance is created by a steering from a diametrical. As it is shown in [16] velocity loss Δv is proportional to the integral:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T (m^2\theta^2 + u^2)dt, \tag{218}$$

where m^2 — weighting coefficient that depends on the form of the ship corpse.

In autosteerages that have been conventionally used a control law is used:

$$u = -(k_1 D + k_0 + \frac{k_u}{D})\theta, \tag{219}$$

where k_1, k_0 and k_u — constant coefficients and the third coefficient is much less than the first two coefficients. On the ship dynamics it does not essentially influence. This coefficient is intended for the compensation of components in perturbing interaction that change very slowly.

In a general case control law (219) does not secure the minimum of velocity loss (as can be easily checked). It is proportional to the integral (218). And therefore already at the end of the sixties other control laws were proposed that will help to decrease velocity loss during the ship movement and in conditions of rough sea. Thus these new laws will give great economic effect.

It is not difficult to find these laws: it is sufficient to introduce new variables $x_3 = \theta$ and $x_4 = \dot{\theta}$. Then equations (214) and (216) can be written in a normal Cauchy form:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -(\alpha^2 + \beta^2)x_1 - 2\alpha x_2 + \eta(t) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\frac{T_2}{T_1^2}x_3 + \frac{1}{T_1^2}x_1 + \frac{1}{T_1^2}u \end{cases} \tag{220}$$

Later a ready apparatus of synthesis theory of optimal control systems can be used. According to it the minimum of integral (218) at the perturbing interaction $\eta(t)$ of "white noise" type will be achieved if we have the following control law:

$$u = -k_1 x_1 - k_2 x_2 - k_3 x_3 - k_4 x_4, \tag{221}$$

where constant coefficients $k_1, \ldots, k_4$ depend on coefficients of a system (220) and on a weighting coefficient m^2 in integral (218).

In order to compute numeric values of coefficients $k_1, \ldots, k_4$ a well-developed compilation of programs for the synthesis theory of optimal regulators can be used. This programs compilation has been naturally developed for the "standard" forms of connection equation — for a normal Cauchy form. If we compute coefficients $k_1, \ldots, k_4$ and if we introduce equation (221) into (220) we shall see that a closed system is stable in a general case and it preserves stability at variations of any ship parameters or a control law. The problem on the synthesis of control law (221) is correctly posed (except some special particular cases).

But it is practically impossible to directly measure and to introduce a moment of perturbing forces $\varphi(t)$ and its derivative in a feedback channel. Therefore it is necessary to exclude redundant variables from equation (221) by means of equivalent transformations. It is not difficult to do so. And after the exclusion we shall come to a control law that easily connects variable θ to be measured and control u. Since transformations that we have used were equivalent this transformed control law must have secured the stability of a closed system and the same value of the loss in (212) as in the law (221). But during marine exposure of a real ship a closed system has shown its instability. This fact has wronged the confidence in optimal control theory for a long period of time and as result — the possibility of application of its results became complicates.

The cause of the paradox was explained in [16]. There for example it was shown that for ships — tankers "Kazbek" (for which equation (214) becomes

$$(690D^2 + 17,2D)\theta = u + \varphi(t), \tag{222}$$

and a weighting coefficient m^2 from the integral (218) is equal to $6,25$) a control law that secures minimum velocity loss and if we use variables θ and

u this law can become

$$u=\left(\frac{690D^2+61,2D+2,5}{0,973-0,06D}-690D^2-17,2D\right)\theta. \qquad (223)$$

A characteristic polynom of a closed system is of the form:

$$690\lambda^2+61,2\lambda+2,5 \qquad (224)$$

and it is of Hurvitz type. The closed system is stable. But if parameters of the ship even by infinitely small magnitudes differ from computed values (from $T_1^2=690$ sec^2 and $T=17,2$ sec^2) and a mathematical model of the ship becomes

$$(690D^2\pm\varepsilon_2D^2+17,2D\pm\varepsilon_1D)\theta=u+\varphi(t), \qquad (225)$$

then a characteristic polynom of a closed system is of the form:

$$(\pm\varepsilon_2\lambda^2\pm\varepsilon_1\lambda)(0,973-0,06\lambda)+(690\lambda^2+61,2\lambda+2,5) \qquad (226)$$

and now a characteristic polynom at infinitely small parameters variations ceases to be of Hurvitz type and a closed system looses stability. Therefore all conclusions about optimal control theory were true. And the mistrust to them was not justified at that time. A closed system (222)–(223) was stable. But from the practical point of view a system that looses stability at infinitely small variations of parameters is equivalent to an unstable system. Here is a paradox.

The same phenomenon (the change of stability preservation property at infinitely small variations of parameters) takes place for other types of ships. This phenomenon also occurs in great many objects of energetics, automated electrodrive, chemical industry etc. — for all objects whose mathematical model reduced to a function from variable x_1 can be turned to a form:

$$A(D)x_1=u+\varphi(t), \qquad (227)$$

where $A(D)$ some degrees polynom from differentiation operator $D=\frac{d}{dt}$ and a perturbing interaction $\varphi(t)$ is a stationary arbitrary process with a spectral density of power of the type (215).

In this case (as it was shown in [16]) a control that helped to secure a minimum to root-mean-square functionals of the kind:

$$J = \lim_{T\to\infty} \frac{1}{T} \int_0^T (m^2 x_1^2 + u^2) dt \tag{228}$$

has the form:

$$u = -\left[\frac{G(D)}{a+bD} - A(D)\right] x_1 \tag{229}$$

where $G(D)$ — degrees polynom of Hurvitz type whose coefficients and coefficients a and b (as well) in a denominator can be computed by a methodics presented in [16]. A characteristic polynom is a closed system (227)–(229) is equal to Hurvitz polynom $G(D)$. The closed system is stable. But if parameters of a control object differ even infinitely little from computed values and if its mathematical model is of the form:

$$A_1(D)x_1 = u + \varphi(t), \tag{230}$$

where $A_1(D) = A(D) \pm \varepsilon_n D^n \pm \ldots \pm \varepsilon_0$ and numbers $\varepsilon_n; \ \ldots \ ; \ \varepsilon_0$ can be infinitely small then a characteristic polynom becomes:

$$(a + b\lambda)(\pm\varepsilon_n \lambda^n \pm \ldots \pm \varepsilon_0) + G(D) \tag{231}$$

and it can be nonHurvitz at infinitely small ε_n. Thus the above problem on the minimum of a functional of the type (228) is not correctly posed. Even if ε_n is infinitely small these functionals can not have a finite value in general.

At the same time if we undertake equivalent (in a classical sense) transformations and if we reduce n-order differential equation (227) to a normal Cauchy form and if we present a perturbing interaction with a spectral density of power (215) as the solution of differential equations system of the form (217) then this problem on functionals (228) minimum will seem to be correctly posed. If we examine the influence of any coefficients variations in a widened system of differential equations (in a normal Cauchy form) on the minimum of functionals (228) we conclude that this minimum will undergo only small changes.

Therefore the phenomenon of changing correctness during equivalent transformations of mathematical models can occur in great many objects of industry, energetics and transport. This phenomenon has been initially found during the examination of optimal system.

For all these objects the check that is made only once is not sufficient. It is necessary to check correctness either by primary, the most close to physical reality equations or we must make sure that transformations of a mathematical model that we apply be equivalent not in a classical sense but in a widened sense as well. The neglect of these recommendations can become the cause of wrecks and catastrophes.

Here is an example from the domain of computation of constant current electrodrives. As it is known an equation of moment equilibrium on the shaft is the main equation of electrodrives:

$$T_M \frac{d\omega}{dt} = M_{mot} - M_r, \tag{232}$$

where T_M — mechanical constant of time, ω — rotation frequency, M_{mot} moment of motor that is proportional to current in armature, M_r — moment of resistance. If resistance moment undergoes oscillations then the velocity of rotation also oscillates. In order to decrease the oscillations the armature current must be effected. On the basis of equation (232) it is easy to write the following equation that reflects these deviations:

$$T_M \frac{dx_3}{dt} = u + \varphi(t), \tag{233}$$

where x_3 — deviation of rotation frequency from equilibrium value, u — controlling interaction, its a deviation of armature current from equilibrium value, $\varphi(t)$ — deviation of resistance moment from its mean value, it's a stationary arbitrary process with a mean value that is equal to zero. Later we shall suppose that it can be represented (as above) in the form of a solution $x_1(t) = \varphi(t)$ of a system of equations (217). We shall put a problem on the synthesis of electrodrive control law that secures the minimum of a functional (228). For simplicity and convenience of checking all further computations let us suppose that $T_M = 1$, $m^2 = 1$, $\beta = 0$, $\alpha = 1$ and then a system of equations (233)–(217) will become:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - 2x_2 + \eta(t) \\ dotx_3 = x_1 + u \end{cases} \tag{234}$$

By means of analytical regulators creation [13] it is not difficult to conclude that the minimum of the above functional can be achieved by means of a regulator:

$$u = -\frac{3}{4}x_1 - \frac{1}{4}x_2 - x_3. \tag{235}$$

It is not difficult to check that a closed system that consists of a control object (234) and a regulator (235) is stable and preserves stability at variations of its parameters. But this conclusion does not correspond to physical reality. Since oscillations of resistance moment x_1 and its derivative x_2 cannot be directly introduced into a feedback channel then equations (234) and (235) must be reduced to variables x_3 and u. Surely we must use here only equivalent transformations. Equation (235) will have the form:

$$(D^2 + 3D + 4)x_3 = (D - 1)u \tag{236}$$

after this transformations.

System (234)–(236) is equivalent (in a classical sense) to system (234)–(235) but contrary to it this system looses stability at infinitely small variations of its some coefficients (at variations of only a certain sign). Just in such a way will a real system behave itself. It is able to loose stability in much unpredicted moments of time when a drift of parameters that is inevitable in the course of exploitation will lead to the change of variations sign. Surely such a system of regulating electrodrive velocity will not work and is dangerous. But we shall not obtain such results if we use conventional methods of checking stability and its preservation at parameters variations.

This example has already been given in [18] on pages 157–158. There the following warnings has been also made: the checking of stability preservation by conventional methods of "conditions spaces" can lead to erroneous results and thus — to wrecks and catastrophes. Sorry to say this warning was not heard. Let us hope that now it will be heard.

§15. How a city where Olympic Games in 2004 will be held was elected.

We have already mentioned that organizations or firms that will apply additional computation checks describes in this book will have great advantages over other firms that do not use them. These supplementary checks will prevent wrecks due to the change of stability resources or the change of correctness. During competitions firms that do not refuse to use the above additional computational checks will have the upper hand over others. This refusal increases the probability of wrecks.

To our regret it is necessary to speak about a negative experience. In 1993–1996 a lot of old auxiliary equipment was changed by new one at Leningrad atomic electric station (LAES) situated near St.Petersburg. Electrodrives, pumps, systems of their control were renewed. St.Petersburg University warned that this new equipment set on such a responsible object as an atomic electric station that is situated near a multi-million city must by all means undergo additional computation check that will help to prevent wrecks. In 1996 there were no doubt that such additional check was extremely necessary. And the University was ready to undertake it. But financing was necessary, a very modes financing of 20 thousand dollars in order that programs compilations be prepared.

But a directorate of LAES as well as the Administration of St.Petersburg governor refused to finance the work. The city's population and especially an organization "the greens" were worried. Several articles in Petersburg newspapers dealing with the problem appeared. There St.Petersburg Administration was criticized for the neglect of the safety of the city.

Just in 1996 the world Olympic Committee discussed a question — where Olympic Games are to be held in 2004. At once several cities declared their wish to take part in the competition. Among them St.Petersburg and Stockholm wanted that this honorable and profitable obligation be laid on them. Struggle competition arose. In the course of this competition representatives of Sweden successfully used the refusal of Petersburg Administration to react to serious warning of its University and its refusal to fulfill their simple methods of increasing the security of LAES. The Swedes were acquainted with the situation around LAES pretty well. They informed Olympic Committee about it and indicated that St.Petersburg whose Administration is able not to react to serious warning of its own University is a dangerous city. In such a city everything can occur and therefore Olympic Games can not be held there. Petersburg Administration knew about this demarche of the Swedes but did nothing. The Swedes had all grounds to say this.

Although if even the beginning of the work in additional check of the equipment of LAES according to the University methods would take place St.Petersburg may sharply increase its chances. But nothing was done and therefore on the 7th of March 1997 during the sitting of the International Olympic Committee at Lauzanne the application of Petersburg was refused on the first round as it is not "safe". But Stockholm successfully overcame the first round and only later Athens was preferred to other cities as an Olympic Games city.

When later Petersburg expenses on advertising the possibility of becoming a city where Olympic Games can be held (advertisements in the streets, gathering of signatures in the favor of Olympic Games etc.) were calculated it turned out that they reached 129 billions rubles (or 21 million dollars according to the exchange course of that time). Petersburg Administration wasted 21 million dollars from its budget and lost the campaign. But it refused to spend 200 thousand dollars for the securing of stability reserves by means of additional checks and creating programs compilations. The city also lost great profits from Olympic Games to be held and Petersburg lost international prestige as it was declared "unsafe". This fact can lead to the decrease of tourists, of international investors etc.

Such is the cost of lack of respect to science. Science can improve the well-being of its people, can improve the level of safety. But in order that the well-being and safety be really improved it is necessary to react to the recommendations of science and to fulfill them in practice.

§16. On continuous dependence of differential equations solutions on parameters.

The material of previous sections allows us to make one more important fact: for many systems of simple differential equations a known theorem on a continuous dependence of solutions on parameters is not true. But at the same time just this theorem lies on the foundation of all practical applications of differential equations theory. Besides it is a basis for a majority of technical computations. It is also a basis for mathematics applications to economy, biology, medicine etc.

In fact we have already pointed out that almost always coefficients and parameters in mathematical models of real objects can not exactly correspond to computed values and almost always they do not remain constant in the course of exploitation. Parameters variations, their small changes are inevitable. But if there is no continuous dependence of solutions on parameters then their infinitely small variations can lead to big changes of solutions. And then computation results are not at all trustworthy.

Therefore a theorem on a continuous solutions dependence on a parameter is given and is proved with the help of a simple differential equations theory in the majority of text-books. But note that this theorem is proved either for one differential equation or for a system of equations written in a normal Cauchy form. It is proved that if right sides of equations systems written in a normal form satisfy Lipshitz conditions (in practice these conditions are almost always satisfied) then solutions of a system in the time interval $0 \leq t \leq T$ depend continuously on parameters (see for example a book by Matvyeyev N.M. "Integration methods of simple differential equations", M., Vissaya shkola (High school), 1967, pp. 259-267). For a system of equations that has not been written in a normal Cauchy form no proof of a theorem on a continuous dependence of solutions on parameters is given. But since it is known that in practice any system of differential equations can be reduced to a normal form by equivalent transformations up to now it was silently supposed that the same continuous dependence on a parameter is satisfied for any systems.

In truth it is not so. Let us examine a known system (100). It is a system of two simple linear differential equations with constant coefficients in relation to two sought for functions x_1 and x_2 with a parameter m. We can easily prove that for this system solutions dependence on a parameter m in a point $m = 1$ is broken. Really a characteristic polynomial of this system is of the form (101). And if for example $m = 1 + \varepsilon$, where ε — is infinitely small positive number then a polynomial (101) acquires a positive big root. And this means that in the solution of the system (100) a quickly increasing exponential member appears. It increases like $e^{\frac{t}{\varepsilon}}$. Even for moderate values of time t solutions of system (100) essentially differ from its solutions for $m \leq 1$ if $m > 1$ and when it is near to 1. At the point $m = 1$ solutions dependence on a parameter m is broken and a point $m = 1$ is peculiar.

At the same time if we reduce system (100) to a normal Cauchy form (when $m = 1$) by equivalent (in a classical sense) transformations then a peculiar role of point $m = 1$ will not be already seen. This fact must not surprise us as we know that transformations that are equivalent in a classical sense but not in a widened sense can (without changing solutions themselves) change for example such their property as stability preservation of solutions at parameters variations.

Since stability reflect properties of solutions (if $t \to \infty$) then in this paragraph we show that transformations that are equivalent in a classical sense but not in a widened one can change some properties of solutions not only when $t \to \infty$ but at a finite time interval, for $0 \leq t \leq T$.

Now let us consider practical consequences. Let us suppose that during the projection of some technical system or a equipment we have to find a solution of system (100) for $m = 1$ or for m that are near to 1. It is not difficult to find a solution but a computation result will not be trustworthy since even during infinitely small deviations (that are inevitable in practice) of real values in parameter m from computed ones the behavior of a system can essentially change. But during the analysis of a mathematical model that is reduces to a normal Cauchy form by means of equivalent transformations we can notice nothing of the kind.

Thus we have set that without the analysis of differences between transformations that are equivalent in a classical sense but not in a widened one and transformations that are equivalent in a widened sense not only stability computations but results of any computations that apply differential equations are not trustworthy. In order to secure trustworthiness additional computations are necessary. Besides the check of used transformations for the equivalence in a widened sense is also necessary.

Now let us examine the question of determining most dangerous equations where continuous dependence of solutions on a parameter is destroyed. First of all let us examine a system of two linear equations with constant coefficients for variables x_1 and x_2. Such a system is of the form:

$$\begin{cases} A_1(D)x_1 = A_2(D)x_2 \\ A_3(D)x_1 = A_4(D)x_2 \end{cases} \tag{237}$$

where $A_1(D), \ldots, A_4(D)$ — polynomials of differentiation operator $D = \frac{d}{dt}$. Their degrees are correspondingly: n_1, n_2, n_3, n_4. A characteristic polynomial in this system is equal to the following determinant:

$$\begin{vmatrix} A_1(D) & -A_2(D) \\ A_3(D) & -A_4(D) \end{vmatrix} = A_2(D)A_3(D) - A_1(D)A_4(D) \tag{238}$$

and at once it is seen that if an equality $n_2 + n_3 = n_1 + n_4$ is satisfied in a characteristic polynomial the reduction of eldest members is possible. This reduction can disappear during infinitely small variations of parameters. But if there is no reduction then a solution of a system will essentially change (since an order will change). And this means that there can be no continuous dependence of solutions on a parameter if $n_2 + n_3 = n_1 + n_4$.

For equations and systems of equations that can be found in modern text-books inequalities $n_1 > n_2$; $n_3 > n_4$ are observed, i.e. in the first equation an order of variable x_1 derivative is larger than an order of variable x_2 derivative. In the second equation an order in a derivative of a variable x_2 is larger than an order in a derivative of a variable x_1. Such equations are called canonical. Equations for which such inequalities are not observed are called non-canonical. In system (100) the first equation is canonical and the second one — non-canonical. An equation (63) from system (62)–(63) will be non-canonical if we consider a control u as the third variable x_3.

In textbooks of Russia non-canonical equations were probably examined for the last time in 1927 in the following book by V.A. Steklov: "Foundations of integration theory in simple differential equations", M.–L., GIZ, 1927. Later only systems of canonic equations were considered since it is simpler to investigate such systems. But at the same time any system that contains non-canonic equations in it can be reduced to a system of canonical equations by means of equivalent transformations. The simplest method is to reduce a system to a normal Cauchy form. A normal form apriori consists of canonical equations since in the left sides first order derivatives stand ($\dot{x}_1$, $\dot{x}_2$, ... , $\dot{x}_n$) and in the right sides — zero order derivatives — variables x_1, x_2, ... , x_n. In right sides there are no derivatives.

Since by equivalent transformations it is possible to reduce a system in which there are non-canonical equations to a more simple system consisting of only canonical equations then after 1927 non-canonical equations gradually disappeared from mathematicians investigations. But just in non-canonical systems an important theorem on a continuous dependence of solutions on parameters is often broken. Really if in a system of the type (237) $n_1 > n_2$ but $n_3 < n_4$ then an equality $n_1 + n_4 = n_2 + n_3$ is quite possible (just this fact takes place in a system (100). And a theorem on a continuous dependence of solutions on parameters is not already satisfied. Here is a very simple example. Let us examine a system with a parameter m:

$$\begin{cases} \ddot{x}_1 = -\dot{x}_2 - x_2 \\ \dot{x}_2 = -m\ddot{x}_1 + e^{-t} \end{cases} \tag{239}$$

and with zero initial conditions: $x_1(0) = \dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$.

If we exclude a variable x_1 from a system (239) by equivalent transformations (in this example it is not difficult to do so) then we shall obtain the following equation for a variable x_2:

$$(1-m)\dot{x}_2 - mx_2 = e^{-t} \tag{240}$$

for which (if we take into consideration an initial condition $x_2(0) = 0$) we obtain the following solution:

$$x_2 = e^{\frac{m}{1-m}t} - e^{-t}. \tag{241}$$

From formula (241) it is at once seen that at a point $m = 1$ a break in solution dependence on a parameter m for any t: $0 \le t < \infty$ occurs. Near $m = 1$ a separation between solutions that (for example) correspond to $m = 0,9999$ and $m = 1,0001$ rashly increases even for small t. So even for $t = 0,01$ (if $m = 0,999$) we shall have $x_2(0,01) = e^{9,99} - e{-}0,01 = 21814$ but when $m = 1,001$ for the same $t = 0,01$ we shall have: $x_2(0,01) = e^{-10,01} - e0,01 = -0,99$.

System (239) can be reduced to a normal Cauchy form by means of introducing a new variable $x_3 = \dot{x}_1$:

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = \frac{m}{1-m}x_2 + \frac{1}{1-m}e^{-t} \\ \dot{x}_3 = -\frac{1}{1-m}x_2 - \frac{1}{1-m}e^{-t}. \end{cases} \tag{242}$$

Formulas (242) allow us to disclose the meaning of the following paradox: in system (242) (according to a general theorem) solutions must continuously depend on parameters but in a system (239) that is equivalent to it there is no continuous dependence. If we denote:

$$\frac{1}{1-m} = n, \tag{243}$$

then system (242) will take the form:

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = \frac{n-1}{n} x_2 + n e^{-t} \\ \dot{x}_3 = -n x_2 - n e^{-t} \end{cases} \tag{244}$$

and its solutions will surely depend on a parameter n continuously. But a parameter n itself (if $m = 1$) "feels" a break as we can see from formula (243).

Thus it must be concluded that for systems of differential equations that contain non-canonical equations the most important theorem on a continuous dependence of solutions on parameters is not true. We must answer the following main question: is the appearance of non-canonical equations possible in primary mathematical models of real technical schemes and equipment (and thus the models that to the greatest extend reflect physical sense)? Partly we have already answered this question in §13. It was shown that system (25)–(26) (where an non-canonical equation (26) is better reflects a physical substance of processes that occur in a real electro-drive than system (33)–(34). Recollect that to system (25)–(26) corresponds a structural scheme shown on fig. 2. But to system (33)–(34) that is equivalent to it corresponds a structural scheme shown on fig. 1. And it does not reflect the physical sense of forming a real inverse connection.

Here are additional examples. Let us examine electro-motors and generators of electric energy that are set on moving object — on ships and aircrafts. As for electro-motors so for generators there is the most important equation in a mathematical model. It is an equation of moments equilibrium on a shaft. Here it is:

$$J\frac{d\omega}{dt} = M_{mov} - M_{res}, \tag{245}$$

where J — inertia moment of rotating masses, ω — frequency of shaft rotation, M_{mov} — moving moment, M_{res} — resistance moment. On a ship resistance moment depends on rolling, i.e. on an angle acceleration of the hull from the second derivative of a heel angle, finally. If we denote by θ a changing heel angle in the ship then a resistance moment will depend on the second derivative of a variable θ, on a value $\ddot{\theta}$. And the most important equation — equation (245) for ships (and for aircrafts as well) will be non-canonical. Thus we can finally conclude that such mathematical models in which continuous dependence of solutions on parameters is destroyed occur rather often in technique. But for all similar mathematical models results of any computation during which differential equations are used are not trustworthy. They are not trustworthy without additional computations examined in §13. Unreliability of computation results can become a cause of serious wrecks and even catastrophes. Above we have shown it.

At the same time additional checks (that we have already examined) guarantee stability preservation during variations of parameters. At the same time they guarantee the preservation of continuous solutions dependence on parameters — at any rate for linear equations with constant coefficients with which we most often encounter in practice. As to nonlinear equations a separate additional investigation must be carried out.

§17. Conclusion

In the conclusion we shall enumerate the main positions of the book (in short):

1. An interesting phenomenon has been found: the possibility of correctness change in a mathematical model during equivalent (in a classical sense) and widely-used transformed equations.

If we speak about systems of differential equations that are mathematical models of many important systems and equipment this phenomenon appears to be the possibility of unexpected change of such an important property of a system as preservation (or not preservation) of stability at coefficients variations after widely-used transformations of equations.

Therefore the existence of a third class of problems in mathematics, physics and technique. These problems are intermediate between known classes of correctly and incorrectly posed problems.

2. Some unexpectedness of this phenomenon is connected with the following: it relates to a such well- known section of mathematics as a theory of equivalent transformations. The development of this theory was finally completed by L. Euler in the 18th century.

3. A practical importance of results stated in this book is in the following: first of all they can help to discover the cause of erroneous conclusions about the preservation of stability. Thus it was possible to destroy one of sources of wrecks and catastrophes in different technical systems and objects.

Besides the finding out of the third class of problems in mathematics, physics and technique are able to change correctness during equivalent transformations of equations. During them it became possible to find out and remove one of causes of errors in computations.

We can't say that the examination of the third class of problems in mathematics, physics and technique is at all completed. Only first steps have been made, interesting laws were deduced, but by all means it is necessary to continue investigation in this field.

Literature

1. *Petrov Yu.P.* Synthesis of optimal control systems when perturbing forces are not completely known. Izdatelstvo Leningrad State University, 1987, 287 pp.

2. *Petrov Yu.P.* Computation of control systems that preserve stability during variations of parameters. St. Petersburg, 1992, 35 pp.

3. *Petrov Yu.P.* Stability of linear systems at variations of parameters. Automatika i Telemekanika (Automatics and Telemechanics), 1994, N 11, pp. 186–189.

4. *Petrov Yu.P.* On hidden dangers in conventional methods of stability check. Izvestiya VUZ, Electromekanika (High School Informations, Electromechanics), 1991, N 11, pp. 106–108.

5. *Petrov Yu.P.* Prevention of wrecks in control systems. Izvestiya VUZ, Electromekanika (High School Informations, Electromechanics), 1994, NN 1–2, pp. 37–40.

6. *Zubov V.I.* Luapunov methods and their application. Izdatelstvo Leningradskogo Universiteta (Leningrad University Edition), 1957, 241 pp.

7. *Zubov V.I.* Mathematical methods of examining systems of automatic regulations. L., Mashinostroyeniye (L., Machines constructions), 1974, 495 pp.

8. *Zubov V.I.* Lections on control theory. M., Nauka (M., Science), 1975, 495 pp.

9. *Barbashin Ye.A.* Luapunov functions. M., Nauka (M., Science), 1970, 240 pp.

10. *Vorotnikov V.I.* Stability of dynamic systems in relation to a part of variables. M., Nauka (M., Science), 1991, 284 pp.

11. *Chang Sh.* Synthesis of optimal automatic control systems. M., Mashinostroyeniye (M., Machines constructions), 1964, 440 pp.

12. *Marriem C.* Theory of optimization and computation of control systems with a feedback. M., Mir (World), 1967, 549 pp.

13. *Letov A.M.* Dynamics of flight and the control. M., Nauka (M., Science), 1969, 360 pp.

14. *Larin V.B., Naumenko K.I., Suntzev B.N.* Synthesis of optimal linear systems with a feedback. Kiev, Naukova Dumka, 1973, 150 pp.

15. *Nadezdin P.V.* On the loss of stability during elementary transformations of differential equations for control systems. Automatika i Telemekanika (Automatics and Telemechanics), 1973, N 1, pp. 185–187.

16. *Petrov Yu.P.* Optimization of control systems that undergo the influence of wind and rough sea. L., Sudostroyeniye (Shipbuilding), 1973, 216 pp.

17. *Petrov Yu.P.* Variational methods in optimal control theory (second edition). L., Energiya (Energetics), 1977, 280 pp.

18. *Abdullayev N.D., Petrov Yu.P.* Theory and methods of projection of optimal regulators. L., Energoatomizdat (Energy-atomic publishers), 1985, 240 pp.

19. *Kharitonov V.L.* On asymptotic stability of equilibrium position in a family of linear differential equations. Differential equations, 1978, N 11.

20. *Polyak B.T., Tzipkin Ya.Z.* Frequency criteria of robust stability and aperiodicity of linear systems. Automatika i Telemekanika (Automatica and Telemechanics), 1990, N 9.

21. *Ivanov V.K., Vassin V.V., Tanava V.P.* Theory of linear noncorrectly posed problems and its applications. M., 1978.

22. *Tikhonov A.N., Arsenin V.Ya.* Solution methods of noncorrect problems. M., 1979, 255 pp.

23. *Willkinson J.H.* Algebraic problems of principal values. M., Nauka (Science), 1970, 564 pp.

24. *Ikramov H.D.* Nonsymmetrical problem of principal values. M., Nauka (Science), 1991, 240 pp.

25. *Ignatiev M.B.* Holonomic automatic systems. Izdatelstvo AN USSR (Edition of USSR Academy of Sciences), 1963, 204 pp.

26. *Gaiduk A.R.* On the investigation of linear systems stability. Automatika i Telemekanika (Automatics and Telemechanics), 1997, N 3, pp. 153–160.

27. *Gaiduk A.R.* Synthesis of control systems when objects are complete under slight conditions. Automatika i Telemekanika (Automatics and Telemechanics), 1997, N 4, pp. 133–144.

28. *Petrov Yu.P.* A mathematical model and physical reality. St.Petersburg, 1997, 58 pp.

29. *Petrov Yu.P., Chervyakov V.V.* Stability systems in boring ships (second edition). St.Petersburg State University Edition, 1997, 261 pp.

30. *Andronov A.A., Vitt A.A.* Oscillations theory. M., Nauka (Science), 1981, 568 pp.

31. *Petrov Yu.P.* Three assays on the history of optimization and optimal control. St.Petersburg, St.Petersburg State University, 1998, 53 pp.

32. *Petrov Yu.P.* The third class of problems in physics and technique — intermediate between correctly and incorrectly posed problems. St.Petersburg, St.Petersburg State University, 1998, 30 pp.

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